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MASTER THESIS

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Weighted Clones

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Abstract: In this thesis we fully describe the structure of all binary parts of weighted clones over the Boolean clones generated by one of the semilattice operations and one or two of the constant operations. We also give a complete description of all atomic and maximal weighted clones over these clones.

Keywords: Relational clones, VCSP, Weighted clones

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Abstract: V této práce kompletně popisujeme strukturu všech binárních částí vážených klonů nad booleovskými klony generovanými jednou z polosvazových operací a jednou nebo dvěma konstantními operacemi. Rovněž poskytujeme úplný popis všech atomických a maximálných vážených klonů nad těmito klony.

Klíčová slova: Relační klony, Vážené klony, VCSP

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Contents

Introduction

The theory of CSP provides an universal apparatus and a simple formal framework for the representation and solution of a wide range of natural combinatorial problems.

The constraint satisfaction problem (CSP) is a computational problem that is in finding an assignment of values to a set of variables, such that this assignment satisfies some specified feasibility conditions. Feasibility conditions of CSP can be parametrized by a set of relations on a suitable domain, so called constraint language. It turns out (see [1]) that instead of vast variety of languages, one can consider CSPs over larger sets of relations, those containing the binary equality relation and closed under primitive positive definitions (so called relational clones), since such a closure does not increase the complexity of CSP problems. Moreover, it was proved by Geiger [2] and V.G. Bondarchuk et al. [3] that there is a one-to-one correspondence between relational clones and sets of operations called function clones, or simply clones, so one can represent and describe any relational clone using operations. This alternative way is very useful, since, for example, all the clones on the Boolean (i.e., two-element) domain are known from the work of E. Post [4].

The generalization of CSP problem, that includes optimization problem, is valued constraint satisfaction problem (VSCP) [5], D. A. Cohen, M. C. Cooper, P. Creed, P. G. Jeavons and S. Živný [6] introduced the concepts of weighted relational clones and weighted clones, that for VCSP play the same role as relational clones and clones for CSP, and proved a one-to-one correspondence between these structures.

Although the complexity of VCSP is now fully understood [7], the structure of weighted clones on the Boolean domain is far from being well understood. The first systematic steps in this direction were made by Jiří Vančura in his thesis [8]. He presented a complete classification of weighted clones over all minimal clones of the Post lattice.

In this thesis we continue in the effort to characterize weighted clones over the clones of the Post lattice. We introduce a concept of binary weighted clones and give a full description of the binary weighted clones over the clones generated by one of the semilattice operations and one or two of the constant operations. To obtain this result we employ a correspondence between binary weighted clones and certain convex sets in a 2-dimensional or 3-dimensional vector space over the rationals. We also give a complete classification of all atomic and maximal weighted clones over these clones.

1. Clones

In this chapter we describe the constraint satisfaction problem (CSP) and explain the correspondence between such problems and clones. Some definitions, examples and results are adapted from [1], [8], [9] and [10].

1.1 CSP and Relational clones

Notation 1. *For any set D and any natural number n, we denote by D^n the set of all n-tuples of elements of D. Any subset of D^n is called an n-ary relation over D. The set of all finitary relations over D is denoted by $\mathbf{R}_D$.*

Definition 1 (CSP)**.** An *instance of the constraint satisfaction problem* (CSP) is a triple $\mathcal{P} = (V, D, C)$ with

- V a nonempty, finite set of variables,
- D a nonempty, finite domain, i.e. set of values,
- C a finite set of constraints, where each constraint is a pair $c = (x, R)$ with
  - x a n-tuple of distinct variables, called the scope of c, and
  - R an n-ary relation on D, called the constraint relation of c.

The decision problem for CSP asks whether there exists a *solution* to $\mathcal{P}$, that is, a function $f : V \to D$ such that, for each constraint $c = (x, R) \in C$, the tuple $f(x)$ belongs to R.

In a fixed-template CSP we fix a domain and a set of allowed constraints.

Definition 2. A *constraint language* Γ is a set of relations on a finite set D. The *constraint satisfaction problem* over Γ, denoted CSP(Γ), is the subclass of the CSP defined by the property that any constraint relation in any instance must belong to Γ.

Various combinatorial problems can be expressed in terms of CSP over a suitable language.

Example 1 (3SAT)**.** *An instance of the standard NP-complete problem [11], 3SAT, is a Boolean formula in conjunctive normal form with exactly three literals per clause. For example, the formula,*

$$\phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_4 \vee x_5 \vee \neg x_1) \wedge (\neg x_1 \vee \neg x_4 \vee \neg x_3)$$

is a satisfiable instance of 3SAT. (Any assignment making x_1 and x_2 false, satisfies ϕ.) 3SAT is equivalent to CSP(Γ_{3SAT}), where $D_{3SAT} = \{0, 1\}$ and $\Gamma_{3SAT} = \{S_{ijk} : i, j, k \in \{0, 1\}\}$, where $S_{ijk} = \{0, 1\}^3 \setminus \{(i, j, k)\}$.

For example, the above formula ϕ corresponds to the following instance of CSP(Γ_{3SAT})

$$\mathcal{P} = (V = \{x_1, x_2, x_3, x_4, x_5\}, D = \{0, 1\},$$
$$C = \{((x_1, x_2, x_3), S_{010}), ((x_4, x_5, x_1), S_{101}), ((x_1, x_4, x_3), S_{111})\}).$$

Example 2 (Graph Unreachability)**.** *An instance of Graph Unreachability consists of a graph $G = (V, E)$ and a pair of vertices, $v, w \in V$. The question is whether there is no path in G from v to w.*

Graph Unreachability can be expressed as the constraint satisfaction problem instance CSP(Γ_{GU}), where $D_{GU} = \{0, 1\}$ and $\Gamma_{GU} = \{=_{\{0,1\}}, C_0, C_1\}$, $=_{\{0,1\}}$ denotes the equality relation over the set $\{0, 1\}$ and C_0, C_1 are constants.

In order to analyze the complexity of CSP instead of the constraint language it is more convenient to use the relational clones, since they considerably reduces the variety of languages to be studied. Relational clones are defined as follows.

Definition 3 (Relational clone)**.** A set of relations $\Gamma \subseteq \mathbf{R}_D$ is a *relational clone* if it

1. contains the binary equality relation,
2. is closed under primitive-positive definitions, i.e. relations defined by relations from Γ, conjunction and existential quantifier are in Γ.

Example 3. *Let Γ be a relational clone and R_1, R_2 be a binary and a ternary relations from Γ. Then the ternary relation*

$$S(x_1, x_2, x_4) = (\exists x_3)(R_1(x_1, x_3) \wedge R_2(x_2, x_3, x_4) \wedge (x_1 = x_4))$$

is also in Γ [8].

Since the set of all relations $\mathbf{R}_D$ is a relational clone and intersection of any set of clones is a clone we can define closure operator.

Definition 4 (Closure operator)**.** For any set of relations $\Gamma \subseteq \mathbf{R}_D$ we define $RelClone(\Gamma)$ to be smallest relational clone that contains Γ.

It is obvious that we can get the relational clone $RelClone(\Gamma)$ from Γ by adding to Γ all relations that one can define with relations in Γ using pp-definitions.

Example 4. *Consider the Boolean constraint language $\Gamma = \{R_1, R_2\}$, where $R_1 = \{(0, 1), (1, 0), (1, 1)\}$ and $R_2 = \{(0, 0), (0, 1), (1, 0)\}$. It is straightforward to check that every binary Boolean relation can be expressed by a pp-formula involving R_1 and R_2. For example, the relation $R_3 = \{(0, 0), (1, 0), (1, 1)\}$ can be expressed by the formula $R_3 = (\exists y)(R_1(x, y) \wedge R_2(y, z))$. Hence the relational clone generated by Γ, $RelClone(\Gamma)$, includes all 16 binary Boolean relations. In fact it can be shown that $RelClone(\Gamma)$ consists of precisely those Boolean relations (of any arity) that can be expressed as a conjunction of unary or binary Boolean relations [9].*

The following theorem shows why we can consider relational clones instead of constraint languages.

Theorem 1. *Let Γ and Σ be finite constraint languages with the same finite domain such that $RelClone(\Gamma) = RelClone(\Sigma)$. Then CSP($\Gamma$) and CSP($\Sigma$) are polynomial-time equivalent.*

In other words, the replacing languages with relational clones does not increase the complexity of problem.

1.2 Clones

For many relational clones it is very hard to describe their structure in terms of relations. However, it turns out that one can represent and describe any relational clone using operations.

Definition 5. For any domain D and any natural number n a mapping $f : D^n \mapsto D$ is called an n-ary operation on D.

Notation 2. *For any finite domain D we denote by $\mathbf{O}_D$ the set of all operations over D. For a natural number k we denote by $\mathbf{O}_D^k \subseteq \mathbf{O}_D$ the set of all k-ary operations over D.*

Notation 3. *Let $i \leq k$ be a natural number. We denote by π_i^k the k-ary projection on the i-th coordinate, i.e., $\pi_i^k(x_1, ..., x_k) = x_i$. When there can arise no confusion, we will denote i-th projection simply by π_i.*

Notation 4. *For any k-ary operation $f \in \mathbf{O}_D^k$ and any m-ary operations $g_1, ..., g_k \in \mathbf{O}_D^m$ we denote by $f[g_1, ..., g_k] \in \mathbf{O}_D^m$ the superposition of f with $g_1, ..., g_k$, i.e.:*

$$f[g_1, ..., g_k](x_1, ..., x_m) = f(g_1(x_1, ..., x_m), ..., g_k(x_1, ..., x_m)).$$

Definition 6 (Clone)**.** A set of operation $O \subseteq \mathbf{O}_D$ is a *clone* if it contains all projections and closed under superposition, i.e. for a k-ary operation $f \in O$ and m-ary operations $g_1, ..., g_m \in O$ the superposition $f[g_1, ..., g_k]$ is in O as well.

Example 5. *The set of all operations $\mathbf{O}_D$ over any finite domain D is a clone. Intersection of any set of clones is a clone again.*

Definition 7 (Closure operator)**.** For any set of operation $O \subseteq \mathbf{O}_D$ we define $Clone(O)$ to be the smallest clone containing O.

Example 6. *Let $D = \{0, 1\}$, $O = \{\wedge\}$, then*

$$Clone(O) = \left\{ f : (a_1, ..., a_n) \mapsto \bigwedge_{i \in I, \emptyset \neq I \subseteq \{1,...,n\}, n \in \mathbb{N}^+} a_i \right\}$$

Definition 8 (k-ary part of a clone)**.** Let $C \subseteq \mathbf{O}_D$ be a clone and let k be a natural number, we define the k-ary part C^k of C as a set of all k-ary operations in C.

Example 7. *Let $D = \{0, 1\}$ and $UD = Clone(\{\neg\})$, i.e. UD is a clone generated by the unary negation operation. The k-ary part of this clone contains exactly $2k$ operations: k projections $\pi_i^k, i = 1, ..., k$ and k negations of projections, $\neg_i^k, i = 1, ..., k$.*

We now define the fundamental correspondence between relations and operations.

Notation 5. *For any domain D, for any k-ary operation f and any collection of n-tuples $\bar{a_1}, ..., \bar{a_k} \in D^n$, where $\bar{a_i} = (a_{1i}, ..., a_{ni})$ we denote by $f(\bar{a_1}, ..., \bar{a_k})$ the n-tuple $(f(a_{11}, ..., a_{1k}), ..., f(a_{n1}, ..., a_{nk}))$.*

Definition 9 (Polymorphism)**.** We say that k-ary operation $f \in \mathbf{O}_D$ preservers an n-ary relation $R \in \mathbf{R}_D$ (or f is a polymorphism of R, or R is invariant under f) if $f(\bar{a_1}, ..., \bar{a_k}) \in R$ for all choices of $\bar{a_1}, ..., \bar{a_k} \in R$.

For any given sets $\Gamma \subseteq \mathbf{R}_D$ and $O \subseteq \mathbf{O}_D$ let

$$Pol(\Gamma) = \{f \in \mathbf{O}_D | \; f \; preserves \; each \; relation \; from \; \Gamma\},$$
$$Inv(O) = \{R \in \mathbf{R}_D | \; R \; is \; invariant \; under \; each \; operation \; from \; O\}.$$

The operators Pol and Inv form a Galois correspondence between $\mathbf{R}_D$ and $\mathbf{O}_D$. Actually, relational clones and clones are exactly the closed sets given by this Galois correspondence. The following theorem allows us to work with clones instead of relational clones [2], [3].

Theorem 2 (Galois Connection for Constraint Languages)**.**

1. *For any finite D, and any $\Gamma \in \mathbf{R}_D$, $Inv(Pol(\Gamma)) = RelClone(\Gamma)$.*
2. *For any finite D and any $O \subseteq \mathbf{O}_D$, $Pol(Inv(O)) = Clone(O)$.*

2. Weighted clones

In this chapter we describe a generalization of CSP, valued constraint satisfaction problem (VCSP), that includes optimization problem. Then we define weighted relational clones and weighted clones, which can be used to determine complexity of VCSP problem in the same way as relational clones and clones are used for CSP problem. Some definitions, examples and results are adapted from [8], [10] and [5].

2.1 VCSP and Weighted relational clones

Definition 10 (Weighted relation)**.** For any domain D and any natural number n a function $R : D^n \mapsto \mathbb{Q} \cup \{\infty\}$ is called cost function, or weighted relation on D of arity n.

Weighted relation associates a rational weight with each of the tuples in some subset of D^n. We denote by $\mathbf{wR}_D$ the set of all weighted relations on D.

Definition 11 (VCSP)**.** An *instance of the valued constraint satisfaction problem* (VCSP) is a triple $\mathcal{P} = (V, D, C)$ with

- V a nonempty, finite set of variables,
- D a nonempty, finite domain,
- C a finite set of constraints, where each element of C is a pair $c = (x, R)$ with
  - x a n-tuple of distinct variables, called the *scope* of c, and
  - R an n-ary weighted relation on D called a *constraint relation*.

An assignment for $\mathcal{P}$ is a mapping $s : V \mapsto D$. The cost of an assignment s, denoted $Cost_{\mathcal{P}}(s)$, is given by the sum of the weights assigned to the restrictions of s onto each constraint scope, that is,

$$Cost_{\mathcal{P}}(s) := \sum_{((x_1,...,x_n),R)\in C} R(s(x_1), ..., s(x_n)).$$

If $R(s(x_1), ..., s(x_n))$ is undefined for some $x = (x_1, ..., x_n)$ (i.e. if the $Cost_{\mathcal{P}}(s)$ is $+\infty$), then the assignment s is said to be infeasible and $Cost_{\mathcal{P}}(s)$ is undefined. A *solution* to $\mathcal{P}$ is a feasible assignment with *minimal cost*.

Definition 12. A *valued constraint language* Γ is a set of weighted relations on a finite set D. The *valued constraint satisfaction problem* over Γ, denoted VCSP(Γ), is the subclass of the VCSP defined by the property that any constraint relation in any instance must belong to Γ.

Example 8 (Minimum Vertex Cover)**.** *The Minimum Vertex Cover problem asks for a minimum size set W of vertices in a given graph $G = (V, E)$ such that each edge in E has at least one endpoint in W. Let $D = \{0, 1\}$. We define*

$$R_1(x,y) := \begin{cases} +\infty, & \text{if } x = y = 0, \\ 0, & \text{otherwise} \end{cases}$$

$$R_2(x) := \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x \neq 0 \end{cases}$$

We denote by Γ_{MIN-VC} the constraint language $\{R_1, R_2\}$. A minimum vertex cover in a graph G with set of vertices $V = \{x_1, ..., x_n\}$ corresponds to the set of vertices assigned the value 1 in some minimum cost assignment to the VCSP(Γ_{MIN-VC}) instance defined by

$$Cost_{\mathcal{G}}(s) := \sum_{x_i \in V} R_2(x_i) + \sum_{(x_i,x_j) \in E} R_1(x_i, x_j).$$

The binary constraints ensure that in any minimal cost assignment at least one endpoint of each edge belongs to the vertex cover.

Note that if we restrict $\mathbf{wR}_D$ to only weighted relations with values $\{0, +\infty\}$, we get CSP problems. Thus, VCSP includes CSP and is actually a generalization.

Now we define a weighted relational clone.

Definition 13 (Weighted relational clone)**.** A set of weighted relations $\Gamma \subseteq \mathbf{wR}_D$ is a *weighted relational clone* if it

1. contains the weighted equality relation (weighted binary relation $=_w$ that is 0 if the two values are equal and ∞ otherwise),
2. is closed under nonnegative multiplication and addition of constant,
3. is closed under sum of relations and minimization over arbitrary arguments.

We say that valued language Γ is closed under expressibility.

Example 9. *Let R_1, R_2 be a binary and a ternary weighted relations from a weighted relational clone Γ, let a_1, a_2 be nonnegative rational numbers and let b be a rational number. Then the weighted ternary relation*

$$S(x_1, x_2, x_4) = \min_{x_3}(a_1 R_1(x_1, x_3) + a_2 R_2(x_2, x_3, x_4) + (x_1 =_w x_4) + b)$$

is also in Γ [8].

Definition 14 (Closure operator)**.** For any set of relations $\Gamma \subseteq \mathbf{wR}_D$ we define $wRelClone(\Gamma)$ to be the smallest weighted relational clone that contains Γ.

Again, as in case with constraint languages and relational clones, to investigate a complexity of VCSP problem over valued constraint language it is sufficient to consider weighted relational clone generated by this language.

Theorem 3. *Let Γ and Σ be finite valued constraint language with the same finite domain such that $wRelClone(\Gamma) = wRelClone(\Sigma)$. Then VCSP($\Gamma$) and VCSP($\Sigma$) are polynomial-time equivalent.*

2.2 Weighted clones

Definition 15 (Weighting)**.** We define a k-ary *weighting* of a clone C to be a function $\omega : C^k \to \mathbb{Q}$ such that $\omega(f) < 0$ only if f is projection and

$$\sum_{f \in C^k} \omega(f) = 0. \tag{2.1}$$

We call the value $\omega(f)$ the weight of ω on f.

We denote by $\mathbf{W}_C$ the set of all weightings of C and by $\mathbf{W}_C^k$ the set of k-ary weightings of C.

Remark 1*.* Since a k-ary weighting ω is simply a rational-valued function that satisfies certain inequalities, scaling ω by nonnegative rationals and summing ω with another k-ary weighting τ of the same clone gives a new weighting of that clone.

Remark 2 (Weighting as a linear combination)*.* We will view a k-ary weighting ω of a clone C as a linear combination of k-ary operations $f_1, ..., f_n$ of that clone:

$$\omega = \omega(f_1)f_1 + \omega(f_2)f_2 + ... + \omega(f_n)f_n.$$

for brevity, we will sometimes omit operations with zero weight in such combination.

Remark 3 (Improper and proper weighting)*.* We will also work with functions $\omega : C^k \to \mathbb{Q}$ that satisfy the zero sum condition but assign negative weight to non-projections. We call these functions *improper weightings* and in this context we call weightings *proper weightings.*

The notion of superposition for operations can be extended to weightings in a natural way as follows.

Definition 16. For any clone C, any weighting $\omega \in \mathbf{W}_C^k$, and any $g_1, ..., g_k \in C^m$, we define the superposition of ω and $g_1, ..., g_k$ to be the (possibly improper) weighting $\omega[g_1, ..., g_k] \in \mathbf{W}_C^m$ defined by:

$$\omega[g_1, ..., g_k](f') := \sum_{\substack{f \in C^k \\ f' = f[g_1, ..., g_k]}} \omega(f)$$

If the result of a superposition is a proper weighting, then we call this superposition a *proper superposition.*

Note that the superposition of a projection with projections is again a projection. Thus, the superposition with projections only is always proper, since the negative weights from projections stay on projections.

Example 10. *Let* $D = \{0, 1\}$*,* $C = Clone(\{\wedge, \vee, C_0, C_1\})$*, where* C_0, C_1 *are constant operations, and let* ω *be the binary weighting*

$$\omega = -3\pi_1 - 4\pi_2 + 1 \wedge +3 \vee +2C_0 + 1C_1.$$

Then the superposition

$$\omega[\pi_1,\pi_1] = -3\pi_1[\pi_1,\pi_1] - 4\pi_2[\pi_1,\pi_1] + 1\wedge[\pi_1,\pi_1] + 3\vee[\pi_1,\pi_1] + \\ + 2C_0[\pi_1,\pi_1] + 1C_1[\pi_1,\pi_1] = -3\pi_1 - 4\pi_1 + 1\pi_1 + 3\pi_1 + 2C_0 + 1C_1 = \\ = -3\pi_1 + 2C_0 + C_1$$

is proper, while the superposition

$$\omega[\pi_2,C_0] = -3\pi_1[\pi_2,C_0] - 4\pi_2[\pi_2,C_0] + 1\wedge[\pi_2,C_0] + 3\vee[\pi_2,C_0] + \\ + 2C_0[\pi_2,C_0] + 1C_1[\pi_2,C_0] = -3\pi_2 - 4C_0 + 1C_0 + 3\pi_2 + 2C_0 + 1C_1 = \\ - C_0 + C_1$$

is improper.

We are now ready to define *weighted clones.*

Definition 17 (Weighted clone)**.** A set of weightings $\mathcal{W} \subseteq \mathbf{W}_C$ over a clone C is a *weighted clone* over C if it

1. contains all zero weightings (i.e. weightings $\theta : C^k \to \{0\}$), for every $k \in \mathbb{N}$;
2. is closed under conical combinations of weightings (more precisely, under nonnegative scaling and sum of weightings, i.e. if $\omega_1, \omega_2 \in \mathcal{W}$ and $q_1, q_2 \in \mathbb{Q}_0^+$, then $\theta := q_1\omega_1 + q_2\omega_2$ is also in $\mathcal{W}$ where $\theta(f) = q_1\omega_1(f) + q_2\omega_2(f)$);
3. is closed under proper superposition (more precisely if a k-ary weighting ω is in $\mathcal{W}$, $g_1, g_2, ..., g_k \in C^m$ and $\omega[g_1, ..., g_k]$ is proper, then $\omega[g_1, ..., g_k] \in \mathcal{W}$);

Example 11. *For any clone C the set $\mathbf{W}_C$ containing all weightings of C is a weighted clone. Also, the set $\mathbf{W}_C^0$ containing all zero weightings of C is a weighted clone. We call these weighted clones trivial, and all others nontrivial.*

Example 12. *The intersection of any set of weighted clones is a weighted clone.*

As for clones, we define *closure operator* and *k-ary parts of a weighted clone.*

Definition 18 (Closure operator)**.** For any set of weightings $\mathcal{W} \subseteq \mathbf{W}_C$ of a clone C we define $wClone(\mathcal{W})$ to be the smallest weighted clone containing $\mathcal{W}$.

Fundamental link between weighted relations and weightings can by defined using weighted polymorphism.

Definition 19 (Weighted polymorphism)**.** Let $R : D^r \mapsto \mathbb{Q} \cup \{\infty\}$ be an r-ary weighted relation on some domain D and let ω be a k-ary weighting of some clone of operations $\mathbf{C}$ on the set D. We say that ω is a *weighted polymorphism* of R if, for any $\bar{x_1}, ..., \bar{x_k} \in D^r$ such that $R(\bar{x_i}) < \infty$ for $i = 1, ..., k$ we have

$$\sum_{f \in C^k} \omega(f) R(f(x_{11}, ..., x_{1k}), ..., f(x_{r1}, ..., x_{rk})) \leq 0.$$

If ω is a weighted polymorphism of R, we say that R is improved by ω. For any given sets $\Gamma \subseteq \mathbf{wR}_D$ and $\mathcal{W} \subseteq \mathbf{W}_C$ let

$$wPol(\Gamma) = \{\omega \in \mathbf{W}_C | \; \forall R \in \Gamma \; \omega \textit{ is a weighted polymorphism of } R\},$$
$$Imp(\mathcal{W}) = \{R \in \mathbf{wR}_D | \; \forall \omega \in W \; R \textit{ is improved by } \omega\}.$$

It follows immediately from the definition of a Galois connection that, for any set D, the mappings $wPol$ and Imp form a Galois connection between $\mathbf{W}_C$ and $\mathbf{wR}_D$. This Galois connection for finite sets D is characterized by the following theorem [6], [10].

Theorem 4 (Galois Connection for Valued Constraint Languages)**.**

1. *For any finite D, and any finite $\Gamma \subseteq \mathbf{wR}_D$, $Imp(wPol(\Gamma)) = wRelClone(\Gamma)$.*
2. *For any finite D and any finite $\mathcal{W} \subseteq \mathbf{W}_C$, $wPol(Imp(\mathcal{W})) = wClone(\mathcal{W})$.*

This means that there is one-to-one correspondence between valued languages closed under expressibility and weighted clones. Thus, we can investigate the complexity of VCSP through weighted clones instead of relational weighted clones.

2.3 Properties of weightings

Definition 20 (k-ary part of a weighted clone)**.** Let $\mathcal{W} \subseteq \mathbf{W}_C$ be a weighted clone and let k be a natural number. We define the k-ary part $\mathcal{W}^k$ of $\mathcal{W}$ as a set of all k-ary weightings in $\mathcal{W}$.

Notation 6. *Accordingly, the binary part of the weighted clone $\mathcal{W}$ is a set of all binary weightings in $\mathcal{W}$. We denote it by $\mathbf{BP}(\mathcal{W})$.*

We do not prove the following technical Lemmas 1, 2 and Theorem 5, which we will need further in the proofs of the main results of this work, because one can find the proofs of these facts, for example, in the work of Jiří Vančura [8]. Lemma 1 shows that any conical combination of arbitrary superpositions of a set of weightings can be obtained by taking a conical combination of superpositions of this set with projections, and then taking a superposition of the result. Thus, since the superposition with projections is always proper, any weighting which can be expressed as a conical combination of arbitrary (possibly improper) superpositions can also be expressed as a superposition of a conical combination of proper superpositions. In other words, it allows us to use improper superpositions when generating a weighted clone – as long as the resulting weighting is proper. We state the lemma for the case of conical combination of two weightings but it is clear that this constructions works for any conical combination.

Lemma 1. *Let ω_1, ω_2 be weightings of a clone C of arities k and l respectively. Let $g_1, ..., g_k$ and $h_1, ..., h_l$ be m-ary operations in C and let α_1, α_2 be nonnegative rationals. We define a (possibly improper) weighting*

$$\omega := \alpha_1\omega_1[\pi_1^{k+l}, ..., \pi_k^{k+l}] + \alpha_2\omega_2[\pi_{k+1}^{k+l}, ..., \pi_{k+l}^{k+l}] \tag{2.2}$$

Then

$$\alpha_1\omega_1[g_1, ..., g_k] + \alpha_2\omega_2[h_1, ..., h_l] = \omega[g_1, ..., g_k, h_1, ..., h_l] \tag{2.3}$$

Very often in further proofs we will use the following consequence of Lemma 1 that allows us to generate a weighted clone $W = wClone(\mathcal{W})$ in one step. Namely, it says that when generating a k-ary part of a weighted clone $W = wClone(\mathcal{W})$, we can consider only k-ary superpositions of weightings from $\mathcal{W}$.

Lemma 2. *(Superposition Lemma) Let $\mathcal{W}$ be a set of weightings of a clone C. A k-ary part of the weighted clone $W := wClone(\mathcal{W})$ is equal to the set of all proper weightings of the form*

$$\alpha_1\omega_1[f_{11}, ..., f_{m_11}] + \alpha_2\omega_2[f_{12}, ..., f_{m_22}] + ... + \alpha_n\omega_n[f_{1n}, ..., f_{m_nn}] \tag{2.4}$$

where $\omega_1, ..., \omega_n$ are weightings from $\mathcal{W}$ with arities $m_1, ..., m_n$, $\alpha_1, ..., \alpha_n \in \mathbb{Q}_0^+$ and $f_{11}, ..., f_{m_11}, ..., f_{1n}, ..., f_{m_nn}$ are k-ary operations from C.

Note that the weightings in the conical combination 2.4 may repeat.

Finally, Theorem 5 provides a way to determine if a weighted clone W is trivial or not.

Theorem 5. *(Positive Projection) Let W be a weighted clone over a clone C. If there exist a k-ary weighting $\omega \in W$ such that $\omega(\pi_i^k) > 0$ for some coordinate $i \leq k$, $k \geq 2$ then $W = \mathbf{W}_C$.*

We now introduce a binary relation on the set of all weightings $\mathbf{W}_C$ over a clone C which will help us to distinguish nontrivial weighted clones.

Definition 21. For a k-ary weighting ω and a m-ary weighting τ over a clone C we say that *weighting ω generates weighting τ*, denoted $\omega \to \tau$, if there exist nonnegative rationals $s_1, s_2, ..., s_n$ and m-ary operations $f_{11}, ..., f_{k_11}, ..., f_{1n}, ..., f_{k_nn}$ from C such that

$$\tau = s_1\omega[f_{11}, ..., f_{k_11}] + s_2\omega[f_{12}, ..., f_{k_22}] + ... + s_n\omega[f_{1n}, ..., f_{k_nn}] \tag{2.5}$$

Note that if τ, ω are proper weightings, then $\tau \in wClone(\omega)$.

Lemma 3. *The binary relation $\to$ determines a quasiorder on $\mathbf{W}_C$.*

Proof. Consider weightings over a clone C. It is obvious, that $\omega \to \omega$. Let's prove transitivity, i.e. that if $\tau \to \theta$ and $\theta \to \omega$, then $\tau \to \omega$. Suppose that τ is a k-ary weighting, θ is a m-ary weighting and ω is a p-ary weighting. Since $\tau \to \theta$ then there exist nonnegative rationals $\alpha_1, ..., \alpha_n$ and m-ary operations $f_{11}, ..., f_{k1}, ..., f_{1n}, ..., f_{kn}$ from C, such that

$$\theta = \alpha_1\tau[f_{11}, ..., f_{k1}] + \alpha_2\tau[f_{12}, ..., f_{k2}]... + \alpha_n\tau[f_{1n}, ..., f_{kn}]$$

Analogically, since $\theta \to \omega$, there exist nonnegative rationals $\beta_1, ..., \beta_t$ and p-ary operations $g_{11}, ..., g_{m1}, ..., g_{1t}, ..., g_{mt}$ from C, such that

$$\omega = \beta_1\theta[g_{11}, ..., g_{m1}] + \beta_2\theta[g_{12}, ..., g_{t2}]... + \beta_t\theta[g_{1t}, ..., g_{mt}]$$

Therefore

$$\begin{aligned}\omega &= \beta_1(\alpha_1\tau[f_{11}, ..., f_{k1}] + ... + \alpha_n\tau[f_{1n}, ..., f_{kn}])[g_{11}, ..., g_{m1}] + ...\\ &... + \beta_t(\alpha_1\tau[f_{11}, ..., f_{k1}] + ... + \alpha_n\tau[f_{1n}, ..., f_{kn}])[g_{1t}, ..., g_{mt}] =\\ &= \beta_1(\alpha_1\tau[h_{111}, ..., h_{k11}] + ... + \alpha_1\tau[h_{111}, ..., h_{kn1}]) + ...\\ &... + \beta_t(\alpha_1\tau[h_{111}, ..., h_{k11}] + ... + \alpha_1\tau[h_{1nt}, ..., h_{knt}]),\end{aligned}$$

where $h_{ijr} = f_{ij}[g_{1r}, ..., g_{mr}]$. We see that ω is just a conical combination of superposition of τ. Therefore $\tau \to \omega$ and the binary relation $\to$ is a quasiorder. $\square$

Definition 22 (Equivalent weightings)**.** We say that two weightings ω, τ over a clone C are *equivalent*, denoted $\omega \leftrightarrow_C \tau$, if $\tau \to \omega$ and $\omega \to \tau$.

By Lemma 3 the relation $\leftrightarrow$ is an equivalence relation. Note that two proper weightings $\omega \leftrightarrow \tau$ if and only if $wClone(\omega) = wClone(\tau)$.

3. Structure of weighted clones

A complete description of a lattice of all clones on a two-element domain $\{0, 1\}$ was provided by Emil Post in 1941 [4]. This lattice is depicted on Figure 3.1.

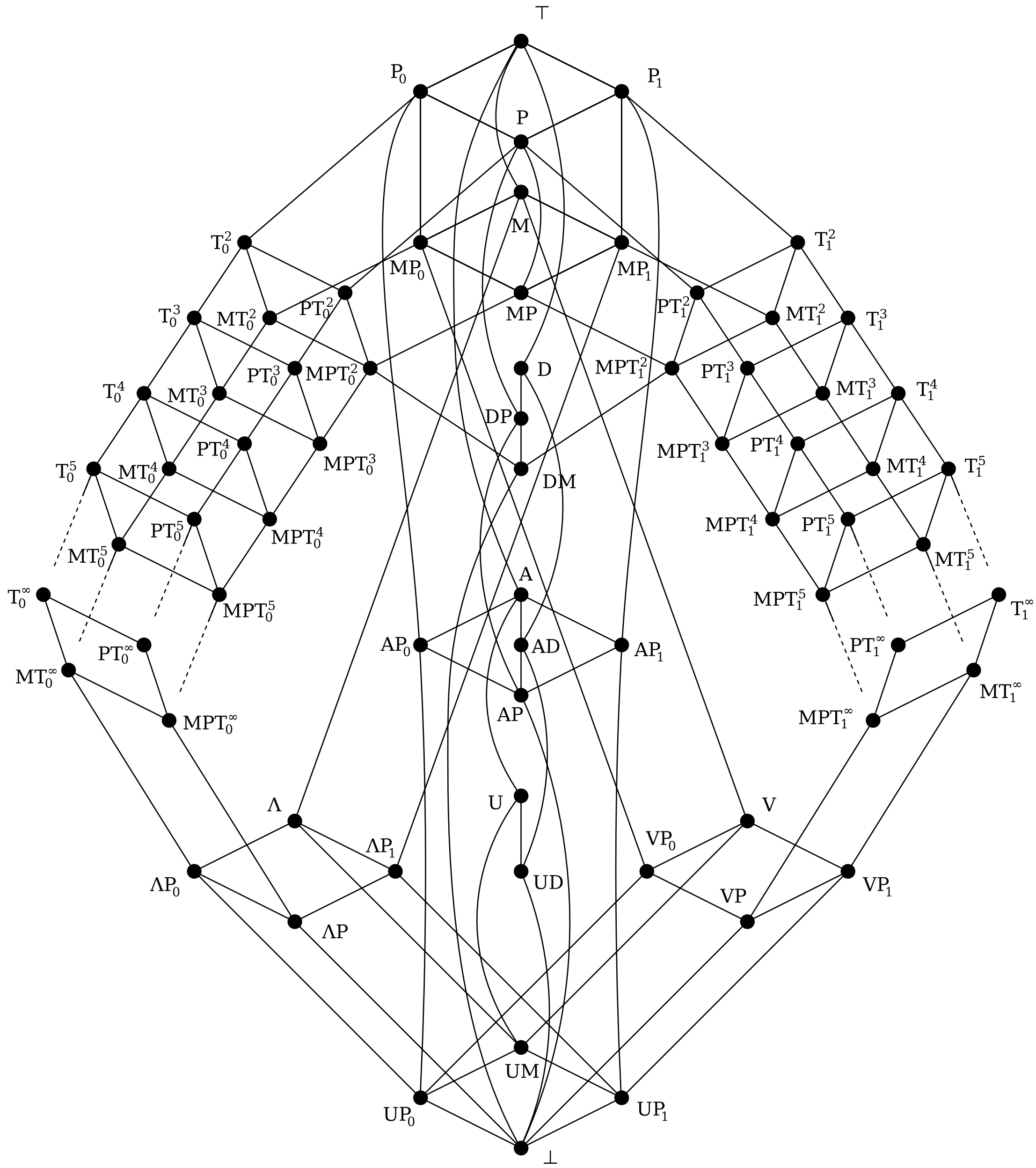

Figure 3.1: The Post lattice of clones on two-element domain. [sourse https://upload.wikimedia.org/wikipedia/commons/thumb/1/19/Post-lattice.svg]

The detailed analysis of the structure of weighted clones over the clones in Post lattice was started by Jiří Vančura [8]. It turned out, that the structure of weighted clones over some clones is quite simple. For example, there is a single nontrivial weighted clone over each of the unary clones $UD = Clone(\{\neg\})$, $UP_0 = Clone(\{C_0\})$, $UP_1 = Clone(\{C_1\})$ and over each of the binary clones

$\wedge P = Clone(\{\wedge\})$, $\vee P = Clone(\{\vee\})$, and there are no nontrivial weighted clones over the ternary clones $AP = Clone(\{min\})$ and $DM = Clone(\{maj\})$. But situation becomes more complicated when dealing with clones generated by more than one operation. Jiří Vančura gave a complete description of the structure of (uncountable many) weighted clones over the unary clone $U = Clone(\{\neg, C_0\})$ and of structure of (three nontrivial) weighted clones over the unary clone $UM = Clone(\{C_0, C_1\})$. He also proved some facts about the clone $MP = Clone(\{\wedge, \vee\})$. We will improve his approach to investigation of the last clone to obtain results for six other weighted clones.

In this chapter we partially describe the structure of weighted clones over the clones $\wedge P_0 = Clone(\{\wedge, C_0\})$, $\wedge P_1 = Clone(\{\wedge, C_1\})$, $\wedge P_{01} = Clone(\{\wedge, C_0, C_1\})$, and dually $\vee P_0 = Clone(\{\vee, C_0\})$, $\vee P_1 = Clone(\{\vee, C_1\})$, $\vee P_{01} = Clone(\{\vee, C_0, C_1\})$. For this first of all we will introduce concepts of binary weighted clone and normed binary weighting. We also give a complete description of all atomic and maximal weighted clones over the above-mentioned clones.

3.1 Binary weighted clones

Definition 23 (Binary weighted clone)**.** *Binary weighted clone* over a clone C is a set of proper binary weightings $W \subseteq \mathbf{W}_C$ such that any proper weighting, which is equal to conical combination of weightings of the form $\omega[f, g]$, where $\omega \in W$ and f, g are 2-ary operations from C, is in W.

Note that due to Lemma 2 the set of binary weightings W over the clone C is a binary weighted clone if and only if it is a binary part of some weighted clone W' over C, $\mathbf{BP}(W')$. It is clear that different binary parts generate different weighted clones, but different weighted clones might have the same binary part.

We will next examine weighted clones over the clones $\wedge P_0 = Clone(\{\wedge, C_0\})$, $\wedge P_1 = Clone(\{\wedge, C_1\})$ and $\wedge P_{01} = Clone(\{\wedge, C_0, C_1\})$. Each of the clones $\wedge P_0$ and $\wedge P_1$ contains exactly 2^k k-ary operations: $2^k - 1$ k-ary meet operations of the form

$$\wedge_I^k(x_1, ..., x_k) := \bigwedge_{i \in I} x_i,$$

where $\emptyset \neq I \subseteq \{1, ..., k\}$, and one constant operation, C_0 or C_1. The clone $\wedge P_{01}$ hence has $(2^k + 1)$ k-ary operations. In our notation $\wedge_I^k$ we will omit the arity k in the superscript when the arity k is clear from the context. For $I = \{i\}$ we simply write $\wedge_i$, or π_i^k, and for $\wedge_{\{1,2\}}$ we write $\wedge$.

We are now not able to describe the complete structure of weighted clones over these three clones, but we will give a full description of their binary parts (or binary weighted clones). Binary operations of the clones $\wedge P_0$, $\wedge P_1$ and $\wedge P_{01}$ are the two projections π_1^2, π_2^2 and $\wedge$, C_0 for $\wedge P_0$, $\wedge$, C_1 for $\wedge P_1$ and $\wedge$, C_0, C_1 for $\wedge P_{01}$.

Definition 24. We say that a k-ary (possibly improper) weighting

$$\omega = \sum_{\emptyset \neq I \subseteq \{1,...,k\}} \omega(\wedge_I^k) \wedge_I^k + bC_0 + cC_1 \tag{3.1}$$

over a clone C is *normed* if $b + c = 1$.

Definition 25. Given a weighted clone W over a clone $C \in \{\wedge P_0, \wedge P_1, \wedge P_{01}\}$ we denote $\mathrm{Norm}(W)$ the set of all normed weighting in W.

Note that every weighting with nonzero weight on at least one constants C_0, C_1 has an equivalent normed form. Since we will further work mostly with binary weighted clones, we define the following weightings.

Definition 26. For non-negative rationals $a_1, a_2, 0 < t < 1$ we denote by $\omega^0_{a_1a_2}$, $\omega^1_{a_1a_2}$ and $\omega^t_{a_1a_2}$ the normed (possibly improper) binary weightings

$$\omega^0_{a_1a_2} = -a_1\pi_1 - a_2\pi_2 + (a_1 + a_2 - 1) \wedge + C_0$$

$$\omega^1_{a_1a_2} = -a_1\pi_1 - a_2\pi_2 + (a_1 + a_2 - 1) \wedge + C_1$$

$$\omega^t_{a_1a_2} = -a_1\pi_1 - a_2\pi_2 + (a_1 + a_2 - 1) \wedge + (1-t)C_0 + tC_1.$$

We denote by $\omega_\wedge$ the binary weighting

$$\omega_\wedge = -a_1\pi_1 - a_2\pi_2 + 2 \wedge .$$

We call the weights on projections of $\omega^t_{a_1a_2}$, i.e. a_1, a_2, the coefficients of $\omega^t_{a_1a_2}$. We will use the notation $\omega_{a_1a_2}$ without superscript when considering weighted clones over the clones $\wedge P_0$, $\wedge P_1$, since there is no danger of confusion.

3.2 Binary weighted clones over the clone with one constant

At first, we consider the weighted clones over the clones $\wedge P_0$, $\wedge P_1$, since they have analogical and easier structure than the common case with both constants. Based on these results we will further describe the structure of binary weighted clones over the clone $\wedge P_{01}$.

To a binary (possibly improper) weighting $\omega_{a_1a_2}$ we assign a point (a_1, a_2) in $\mathbb{Q}^2_{\geq 0}$, where

$$\mathbb{Q}^2_{\geq 0} = \{(a, b) \in \mathbb{Q}^2 : a, b \geq 0\}.$$

Thus, for example, as it is shown in Figure 3.2, a proper weighting $\omega_{2,3} = -2\pi_1 - 3\pi_2 + 4 \wedge + C_0$ corresponds to the point $(2, 3)$, and improper weightings $\omega_{\frac{1}{2},0} = -\frac{1}{2}\pi_1 - \frac{1}{2} \wedge + C_0$, $\omega_{0,0} = -\wedge + C_0$ correspond to points $(\frac{1}{2}, 0)$ and $(0, 0)$ respectively.

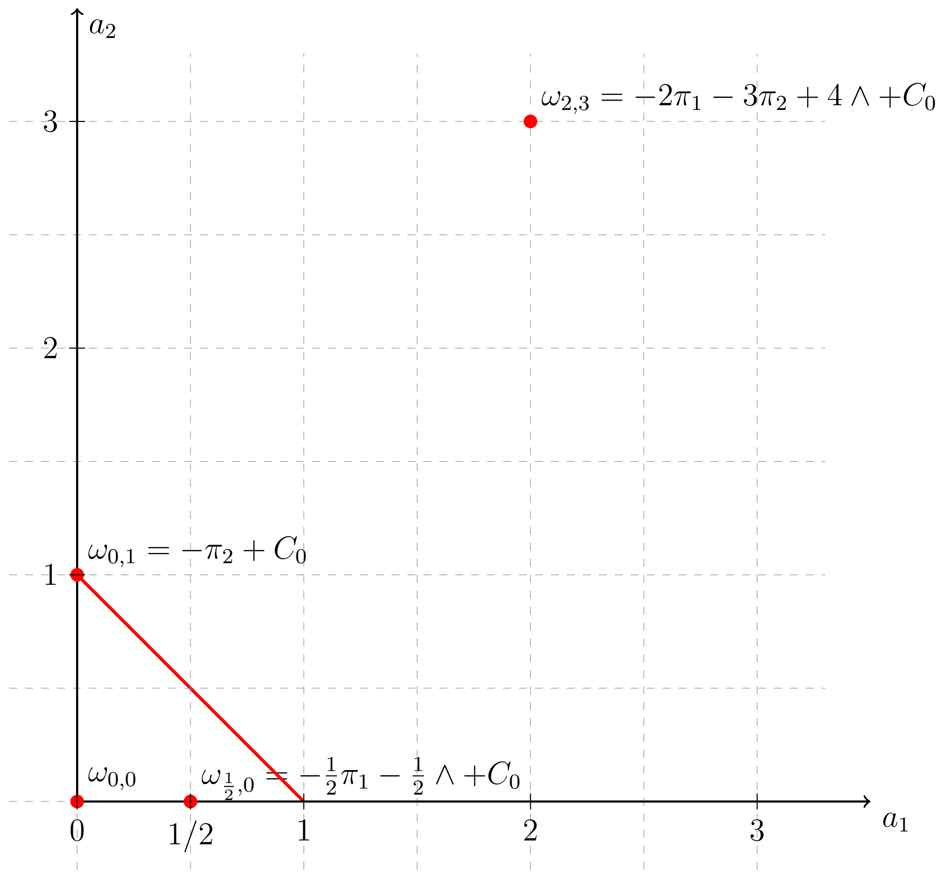

Figure 3.2: Correspondence between normed weightings and points in $\mathbb{Q}^2_{\geq 0}$.

In this way, every set of normed weightings corresponds to a set of points in $\mathbb{Q}^2_{\geq 0}$. We will describe all binary weighted clones via the structure of the corresponding sets in $\mathbb{Q}^2_{\geq 0}$. The critical properties of those sets are described in the following definition.

Definition 27. We say that a subset M of $\mathbb{Q}^2_{\geq 0}$ *satisfies Property (*)* if:

(1) M is convex, i.e., for every $\mathbf{x_1}, \mathbf{x_2} \in M$ and every $t \in [0,1]$ we have $t\mathbf{x_1} + (1-t)\mathbf{x_2} \in M$;

(2) M contains the points $(0,1)$ and $(1,0)$;

(3) M is symmetric with respect to the line $x = y$, i.e. $(x,y) \in M \Leftrightarrow (y,x) \in M$;

(4) if M contains a point (x,y), then $(x,0) \in M$ and $(0,y) \in M$.

For the clones $\wedge P_0$, $\wedge P_1$ we define two objects, $M(W)$ and $W(M)$, as follows:

Definition 28. Given a binary weighted clone W over a clone $C \in \{\wedge P_0, \wedge P_1\}$ we denote by $M(W)$ the following set of points in $\mathbb{Q}^2_{\geq 0}$

$$M(W) := \{(b_1, b_2) : \omega_{b_1 b_2} \textit{ is (possibly improper) weighting such that } \omega_{b_1 b_2} \leftarrow \omega \in W\}.$$

Definition 29. Given a set M of points in $\mathbb{Q}^2_{\geq 0}$ we denote by $W(M)$ the set of proper binary weightings

$$W(M) := \{s\omega_{a_1 a_2} : (a_1, a_2) \in M, s \geq 0, a_1 + a_2 \geq 1\} \cup \{\textit{the zero weighting}\}.$$

3.2.1 Binary weighted clones over the clone $\wedge P_0$

After these preliminary definitions we now start with describing weighted clones over the clone $\wedge P_0 = Clone(\{\wedge, C_0\})$.

Lemma 4. *For every nonnegative rationals a_1, a_2, $b_1 + b_2 > 1$ the following is true over the clone $\wedge P_0$:*

(1) $\omega_\wedge \to \omega_{a_1a_2} \not\to \omega_\wedge$;

(2) $\omega_{a_1a_2} \to \omega_{1,0} \not\to \omega_{b_1b_2}$.

Proof. (1) Consider the binary superposition

$$\begin{aligned}&a_1\omega_\wedge[\wedge,\pi_2] + a_1\omega_\wedge[\pi_1,\wedge] + \omega_\wedge[\wedge, C_0] =\\ &= a_2(-\wedge - \pi_2 + 2\wedge) + a_1(-\pi_1 - \wedge + 2\wedge) + (-\wedge - C_0 + 2C_0) =\\ &= -a_1\pi_1 - a_2\pi_2 + (a_1 + a_2 - 1)\wedge + C_0.\end{aligned}$$

Therefore $\omega_\wedge \to \omega_{a_1a_2}$.

On the other hand, we cannot remove positive weight from C_0 by any of the $4^2 = 16$ binary superpositions of $\omega_{a_1a_2}$. Indeed, there is the trivial superposition $\omega_{a_1a_2}[\pi_1, \pi_2]$, there is the superposition $\omega_{a_1a_2}[\pi_2, \pi_1]$ that switches the weights on projections, there is the zero superposition $\omega_{a_1a_2}[C_0, C_0]$, there are 6 superpositions for $i = 1, 2$

$$\begin{aligned}&\omega_{a_1a_2}[\pi_i,\pi_i] = -\pi_i + C_0,\\ &\omega_{a_1a_2}[\pi_i, C_0] = -a_1\pi_i - a_2C_0 + (a_1 + a_2 - 1)C_0 + C_0 = -a_1\pi_i + a_1C_0,\\ &\omega_{a_1a_2}[C_0,\pi_i] = -a_2\pi_i + a_2C_0,\end{aligned}$$

equal $\omega = -\pi_i + C_0$ for $i = 1, 2$ multiplied by some nonnegative rational, there are 3 superpositions,

$$\begin{aligned}&\omega_{a_1a_2}[\wedge,\wedge] = -\wedge + C_0,\\ &\omega_{a_1a_2}[\wedge, C_0] = -a_1 \wedge - a_2C_0 + (a_1 + a_2 - 1)C_0 + C_0 = -a_1 \wedge + a_1C_0,\\ &\omega_{a_1a_2}[C_0,\wedge] = -a_2 \wedge + a_2C_0,\end{aligned}$$

equal $\eta = -\wedge + C_0$ multiplied by some nonnegative rational. And finally, there are 4 superpositions for $i = 1, 2$

$$\begin{aligned}&\omega_{a_1a_2}[\pi_i,\wedge] = a_1\pi_i - a_2 \wedge + (a_1 + a_2 - 1)\wedge + C_0 = -a_1\pi_i + (a_1 - 1)\wedge + C_0,\\ &\omega_{a_1a_2}[\wedge,\pi_i] = -a_2\pi_i + (a_2 - 1)\wedge + C_0,\end{aligned}$$

which still have positive weight on C_0. Therefore $\omega_{a_1a_2} \not\to \omega_\wedge$.

(2) Consider the binary superposition

$$\frac{1}{a_1}\omega_{a_1a_2}[\pi_1, C_0] = -\pi_1 + C_0.$$

Therefore $\omega_{a_1a_2} \to \omega_{1,0}$.

On the other hand, we cannot get positive weight on $\wedge$ by any of the $4^1 = 4$ binary superpositions of $\omega_{1,0}$, since $\omega_{1,0}[C_0, -]$ is the zero weighting, $\omega_{1,0}[\pi_i, -] = -\pi_i + C_0$ for $i = 1, 2$ and $\omega_{1,0}[\wedge, -] = -\wedge + C_0$. Therefore $\omega_{1,0} \not\to \omega_{b_1b_2}$.

□

We will further use the following important consequence of the proof of the previous lemma.

Corollary 1. *Let $\omega_{a_1a_2}$ be a normed (possibly improper) weighting over the clone $\wedge P_0$ and let*

$$P = \{(a_1, a_2), (a_2, a_1), (a_1, 0), (0, a_1), (a_2, 0), (0, a_2), (0, 1), (1, 0), (0, 0)\}.$$

Then

1. *For each point $(a'_1, a'_2) \in P$ there exist binary operations f, g from $\wedge P_0$ such that $\omega_{a_1a_2}[f, g]$ is a positive multiple of $\omega_{a'_1a'_2}$.*
2. *For each binary operations f, g from $\wedge P_0$, $\omega_{a_1a_2}[f, g]$ is either the zero weighting or a positive multiple of $\omega_{a'_1a'_2}$, where $(a'_1, a'_2) \in P$.*

Lemma 5. *For every nontrivial binary weighted clone W over the clone $\wedge P_0$ the set $M(W)$ satisfies Property $(*)$.*

Proof. Consider a binary weighted clone W that contains an arbitrary nonzero weighting with the zero weight on the constant C_0, say $\omega = -a_1\pi_1 + a_2\pi_2 + (a_1 + a_2)\wedge$. Then

$$\frac{1}{a_1 + a_2}(\omega + \omega[\pi_2, \pi_1]) = -\pi_1 - \pi_2 + 2\wedge = \omega_\wedge.$$

By Lemma 4 we know that $\omega_\wedge \to \omega_{b_1b_2}$ for every $b_1, b_2 \geq 0$. Thus, $M(W) = \mathbb{Q}^2_{\geq 0}$ and therefore $M(W)$ satisfies Property $(*)$.

Now consider a binary weighted clone W whose all nonzero weightings have nonzero weight on the constant C_0. We have to prove that $M(W)$ satisfies Property $(*)$.

We first prove that $M(W)$ is convex. Note that all points in the triangle with the vertices $(1, 0), (0, 0), (0, 1)$ correspond to improper weightings. Consider any point $(a_1, a_2) \in M(W)$. According to Definition 28 there exist proper binary weighting $\omega_{a'_1a'_2}$ such that $\omega_{a'_1a'_2} \to \omega_{a_1a_2}$. For every $a_1, a_2 \geq 0$ the weighting $\omega_{a_1a_2}$ by the superpositions $\omega_{a_1a_2}[\pi_1, \pi_1]$, $\omega_{a_1a_2}[\pi_2, \pi_2]$ and $\omega_{a_1a_2}[\wedge, \wedge]$ generates the weightings $\omega_{1,0}$, $\omega_{0,1}$ and $\omega_{0,0}$ respectively. By the transitivity $\omega_{a'_1a'_2} \to \omega_{1,0}$, $\omega_{a'_1a'_2} \to \omega_{0,1}$ and $\omega_{a'_1a'_2} \to \omega_{0,0}$. Since for every point (a''_1, a''_2), where $a''_1, a''_2 \geq 0$, $a''_1 + a''_2 \leq 1$ there exist nonnegative rationals s_1, s_2, s_3, where $s_1 + s_2 + s_3 = 1$, such that

$$\omega_{a''_1a''_2} = s_1\omega_{1,0} + s_2\omega_{0,0} + s_3\omega_{0,1},$$

then $\omega_{a'_1,a'_2} \to \omega_{a''_1a''_2}$. Hence $M(W)$ contains the whole triangle with the vertices $(1, 0), (0, 0), (0, 1)$.

Now consider any two points $(a_1, a_2), (b_1, b_2) \in M(W)$ such that $a_1 + a_2 \geq 1$, $b_1 + b_2 \geq 1$. Due to definition 28, $\omega_{a_1a_2}, \omega_{b_1b_2} \in \mathrm{Norm}(W)$. For every $t \in [0, 1]$ the point

$$t(a_1, a_2) + (1 - t)(b_1, b_2) = (ta_1 + (1 - t)b_1, ta_2 + (1 - t)b_2)$$

corresponds to the normed weighting

$$\begin{aligned}\omega_{(ta_1+(1-t)b_1),(ta_2+(1-t)b_2)} &= t\omega_{a_1a_2} + (1 - t)\omega_{b_1b_2} = \\ &= -(ta_1 + (1 - t)b_1)\pi_1 - (ta_2 + (1 - t)b_2)\pi_2 + \\ &+ (t(a_1 + a_2) + (1 - t)(b_1 + b_2) - 1) \wedge + C_0.\end{aligned}$$

Since the binary weighted clone W is closed under nonnegative scaling and sum of weightings, then the weighting $\omega_{(ta_1+(1-t)b_1),(ta_2+(1-t)b_2)} \in \mathrm{Norm}(W)$ and therefore the point $(ta_1 + (1-t)b_1, ta_2 + (1-t)b_2) \in M(W)$. Together with the fact that $M(W)$ contains the triangle with the vertices $(1,0), (0,0), (0,1)$ it follows that $M(W)$ is convex.

Now it is sufficient to note that by Corollary 1 for every (possibly improper) normed weighting $\omega_{a_1a_2}$ such that $(a_1, a_2) \in M(W)$, $M(W)$ contains all the points (a'_1, a'_2), where

$$(a'_1, a'_2) \in \{(a_1, a_2), (a_2, a_1), (a_1, 0), (0, a_1), (a_2, 0), (0, a_2), (0, 1), (1, 0), (0, 0)\}.$$

Therefore $M(W)$ satisfies the conditions (2), (3), (4) in the definition of Property $(*)$. □

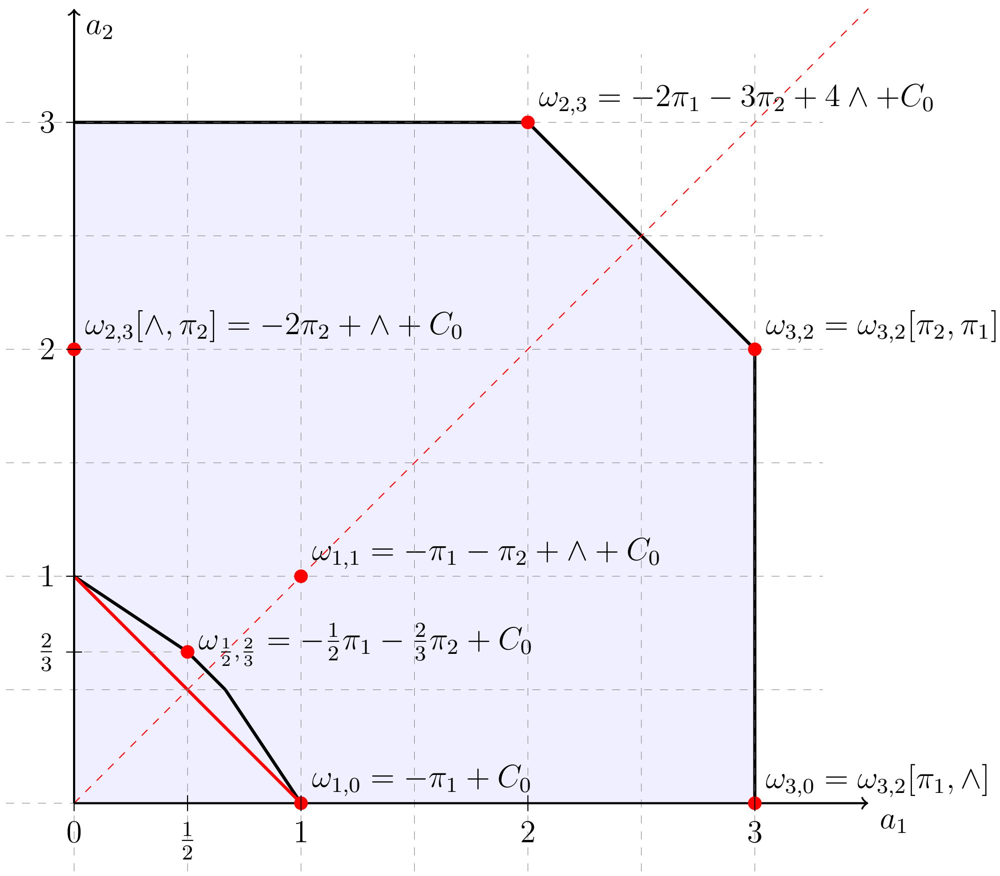

Figure 3.3: Illustration to Lemma 5.

Lemma 6. *For every set $M \subseteq \mathbb{Q}^2_{\geq 0}$ that satisfies Property $(*)$, $W(M)$ is a binary weighted clone.*

Proof. Consider a proper binary weighting ω which is equal to

$$\omega = r_1\omega_1[f_1, g_1] + r_2\omega_2[f_2, g_2] + ... + r_n\omega_n[f_n, r_n],$$

where $r_1, r_2, ..., r_n$ are nonnegative rationals, $\omega_1, \omega_2, ..., \omega_n \in W(M)$ and f_i, g_i for $i = 1, ..., n$ are binary operations from the clone $\wedge P_0$. According to Definition 29, each ω_i for $i = 1, ..., n$ is of the form $s_i\omega_{a_ib_i}$, where $(a_i, b_i) \in M$. By Corollary 1, for every f_i, g_i from $\wedge P_0$, the superposition $\omega_{a_ib_i}[f_i, g_i]$ is either the zero weighting or positive multiple of $\omega_{a'_ib'_i}$ for some $(a'_i, b'_i) \in M$. Assume without loss of generality that $\omega_{a_ib_i}[f_i, g_i]$ for every $i = 1, ..., n$ is not the zero weighting. Therefore, for some positive rationals $k_1, ..., k_n$ we can rewrite ω as

$$\begin{aligned} \omega &= r_1s_1k_1\omega_{a'_ib'_i} + ... + r_ns_nk_n\omega_{a'_nb'_n} = \\ &= (r_1s_1k_1 + ... + r_ns_nk_n)(p_1\omega_{a'_1b'_1} + ... + p_n\omega_{a'_nb'_n}) = s\omega_{ab}, \end{aligned}$$

where $s = (r_1s_1k_1 + ... + r_ns_nk_n)$, $(p_1 + ... + p_n) = 1$, $a = (p_1a'_1 + ... + p_na'_n)$ and $b = (p_1b'_1 + ... + p_nb'_n)$. Since M is convex, then $(a, b) \in M$ and therefore $\omega \in W(M)$. According to Definition 23, $W(M)$ is a binary weighted clone over the clone $\wedge P_0$. □

Before we turn to the main result of this section, which concerns the complete classification of binary weighted clones over the clone $\wedge P_0$, we formulate Theorem 6 about the largest and the smallest nontrivial weighted clones over the clone $\wedge P_0$.

Theorem 6. *The largest nontrivial weighted clone over the clone $\wedge P_0$ is $W_\wedge$, defined as follows: a k-ary weighting ω is in $W_\wedge$ if and only if for each nonempty set of coordinates $I \subseteq \{1, ..., k\}$*

$$\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J) \leq 0. \tag{3.2}$$

Moreover $W_\wedge$ is generated by the weighting $\omega_\wedge = -\pi_1 - \pi_2 + 2\wedge$.

The smallest nontrivial weighted clone over the clone $\wedge P_0$ is $W_{1,0}$, defined as follows: for every k

$$W^k_{1,0} = \{\omega \in W_{\wedge P_0} : \textit{ for every } I \subseteq \{1, ..., k\} \textit{ such that } |I| > 1,\ \omega(\wedge_I) = 0\}.$$

Moreover $W_{1,0}$ is generated by the unary weighting $\omega_{1,0} = -\pi_1 + C_0$.

We do not give a whole proof for the largest nontrivial weighted clone $W_\wedge$, since the main idea belongs to Jiří Vančura and was in detail described by him for the weighted clone over the clone $\wedge = wClone(\wedge)$ [8]. We only consider parts that concern the appearance of the constant C_0 in the clone $\wedge P_0$.

Proof. For the first part of the theorem we first prove that any k-ary weighting ω that is not in $W_\wedge$ generates the trivial weighted clone $W_{\wedge P_0}$. Let $\emptyset \neq I \subseteq \{1, ..., k\}$ be a set of coordinates for which ω violates the condition (3.2), i.e. such that $\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J) > 0$. Assume without loss of generality that $I = \{1, 2, ..., p\}$. We consider a superposition of ω with two projections π_1, π_2 such that we input π_1 into the coordinates from I and π_2 elsewhere. Then the resulting weighting

$$\omega' = \omega[\underbrace{\pi_1, ..., \pi_1}_{p}, \pi_2, ..., \pi_2] =$$

$$\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J)\pi_1 + \sum_{\emptyset \neq J' \subseteq \{1,...,k\} \setminus I} \omega(\wedge_{J'})\pi_2 + \sum_{\substack{\emptyset \neq J'',\ J'' \cap I \neq \emptyset \wedge \\ J'' \cap J' \neq \emptyset}} \omega(\wedge_{J''}) \wedge + cC_0$$

is proper and has weight $\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J) > 0$ on projection π_1. Due to Theorem 5, ω generates the trivial weighted clone $W_{\wedge P_0}$.

We now can prove that $W_\wedge = wClone(\omega_\wedge)$. To show, that $wClone(\omega_\wedge) \subseteq W_\wedge$ we have to prove that $\omega_\wedge$ does not generate any weightings outside of $W_\wedge$. We already know that any weighting outside $W_\wedge$ generates all weightings, i.e. if $\omega_\wedge$ generates some k-ary weighting, then $\omega_\wedge$ generates any m-ary weighting (and due to Lemma 2 we can do it in one step). Thus, it is sufficient to show that we cannot generate any binary weighting outside of the $W_\wedge$. In order to generate positive weight on projection we have to move the weight 2 from $\wedge$ to a projection. But it can be done only by superpositions $\omega_\wedge[\pi_i, \pi_i]$ for $i = 1, 2$, which is the zero weighting.

The fact that every k-ary weighting $\omega \in W_\wedge$, which has the zero weight on the constant C_0, is generated by the weighting $\omega_\wedge$, was proved by Jiří Vančura using Farkas' lemma (see [8]). Thus, to prove that $W_\wedge \subseteq wClone(\omega_\wedge)$ it is now sufficient to show that any nonzero k-ary weighting $\theta = \sum_{\emptyset \neq I \subseteq \{1,...,k\}} \omega(\wedge_I^k) \wedge_I^k + cC_0$ with $c > 0$ from $W_\wedge$ can be generated by an appropriate weighting ω from $W_\wedge$ with the zero weight on the constant C_0. Consider the $(k+1)$-ary weighting

$$\omega = \sum_{\emptyset \neq I \subseteq \{1,...,k\}} \omega(\wedge_I^{k+1}) \wedge_I^{k+1} + c \wedge_{\{1,...,k,k+1\}}^{k+1} .$$

The weighting ω is from $W_\wedge$. Indeed, since $\theta \in W_\wedge$, then for each nonempty set of coordinates $I \subseteq \{1, ..., k\}$ the sum $\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J^k) \leq 0$. Since $\omega(\wedge_{I'}^{k+1}) = 0$ for every proper subset $I' \subseteq \{1, ..., k+1\}$ such that $\{(k+1)\} \in I'$ and the sum $\sum_{\emptyset \neq I \subseteq \{1,...,k\}} \omega(\wedge_I^{k+1}) + c = 0$, then the extension of the set $\{1, ..., k\}$ to the set $\{1, ..., k, k+1\}$ does not change the inequality. Also, it is easy to see that the superposition $\omega[\pi_1, ..., \pi_k, C_0]$ equals θ. Thus, by transitivity, $\omega_\wedge \to \theta$ and $W_\wedge \subseteq wClone(\omega_\wedge)$. Therefore, $W_\wedge = wClone(\omega_\wedge)$ and since each weighting outside of $W_\wedge$ generates the trivial weighted clone $\wedge P_0$, $W_\wedge$ is the largest weighted clone over the clone $\wedge P_0$.

For the second part of the theorem we first prove that $W_{1,0} = wClone(\omega_{1,0})$. On the one hand, each k-ary weighting ω from $W_{1,0}$ is of the form

$$\begin{aligned} &\omega = -a_1\pi_1 - a_2\pi_2 - ... - a_k\pi_k + (a_1 + a_2 + ... + a_k)C_0 = \\ &a_1(-\pi_1 + C_0) + a_2(-\pi_2 + C_0) + ... + a_k(-\pi_k + C_0) = \\ &a_1\omega_{1,0}[\pi_1] + a_2\omega_{1,0}[\pi_2] + ... + a_k\omega_{1,0}[\pi_k]. \end{aligned}$$

Therefore, each weighting $\omega \in W_{1,0}$ is in $wClone(\omega_{1,0})$, i.e. $W_{1,0} \subseteq wClone(\omega_{1,0})$. On the other hand, due to Lemma 2, we can generate a k-ary part of the weighted clone $wClone(\omega_{1,0})$ in one step by only k-ary superpositions of $\omega_{1,0}$. That is, for every proper k-ary weighting $\omega \in wClone(\omega_{1,0})$ there exist nonnegative rationals $a_1, ..., a_k$, b_I for $I \subseteq \{1, ..., k\}, |I| > 1$ and c such that ω is equal to:

$$\omega = +a_1\omega_{1,0}[\wedge_1] + ... + a_k\omega_{1,0}[\wedge_k] + \sum_{\substack{\emptyset \neq I \subseteq \{1,...,k\} \\ |I|>1}} b_I\omega_{1,0}[\wedge_I] + c\omega_{1,0}[C_0] =$$

$$a_1(-\wedge_1 + C_0) + ... + a_k(-\wedge_k + C_0) + \sum_{\substack{\emptyset \neq I \subseteq \{1,...,k\} \\ |I|>1}} b_I(-\wedge_I + C_0) + c(-C_0 + C_0) =$$

$$-a_1 \wedge_1 - ... - a_k \wedge_k - \sum_{\substack{\emptyset \neq I \subseteq \{1,...,k\} \\ |I|>1}} b_I \wedge_I + (a_1 + ... + a_k)C_0 + (\sum_{\substack{\emptyset \neq I \subseteq \{1,...,k\} \\ |I|>1}} b_I)C_0.$$

Since ω is proper, we have $b_I = 0$ for all $I \subseteq \{1, ..., k\}, |I| > 1$. Thus, every k-ary weighting $\omega \in wClone(\omega_{1,0})$ is in $W_{1,0}^k$ and $wClone(\omega_{1,0}) \subseteq W_{1,0}$. Therefore $wClone(\omega_{1,0}) = W_{1,0}$. This proves that $W_{1,0}$ is actually a weighted clone and is generated by the weighting $\omega_{1,0}$.

Finally, we show that $W_{1,0}$ is the smallest nontrivial weighted clone, i.e. is contained in all nontrivial weighted clones. Indeed, let τ be an arbitrary k-ary nonzero weighting that does not generate all weightings. If τ has the zero weight on the constant C_0, then τ generates the largest nontrivial weighted clone $W_\wedge$. Otherwise, if $\tau(C_0) = c > 0$ consider the superposition:

$$\frac{1}{c}\tau[\pi_1, \pi_1, ..., \pi_1] = \frac{1}{c}\Big(c\pi_1 + cC_0\Big) = -\pi_1 + C_0.$$

That is, in both cases $wClone(\omega_{1,0}) \subseteq wClone(\tau)$. Therefore $W_{1,0}$ is smallest weighted clone over the clone $\wedge P_0$. □

The next theorem another significant weighted clone over the clone $\wedge P_0$.

Theorem 7. *The nontrivial weighted clone over the clone $\wedge P_0$ that contains all others nontrivial weighted clones except the weighted clone $W_\wedge$ is $W_{c\neq 0}$, defined as follows: a k-ary weighting ω is in $W_{c\neq 0}$ if and only if for each nonempty set of coordinates $I \subseteq \{1, ..., k\}$ and for each nonempty set of coordinates $T \subseteq \{1, ..., k\}$ such that $T \cap \{j : \omega(\pi_j^k) < 0\} \neq \emptyset$,*

$$\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J^k) \leq 0, \tag{3.3}$$

$$\sum_{\emptyset \neq J \subseteq T} \omega(\wedge_J^k) < 0. \tag{3.4}$$

Proof. We first prove that $W_{c\neq 0}$ is a weighted clone. It is obvious that $W_{c\neq 0}$ is closed under conical combinations of weightings, since for each $\theta, \tau \in W_{c\neq 0}$ and $p, q \in \mathbb{Q}_0^+$ the weighting $\omega = p\theta + q\tau$ has weight on operation f that equals $\omega(f) = p\theta(f) + q\tau(f)$. Thus we only have to prove that $W_{c\neq 0}$ is closed under proper superposition. Consider a proper k-ary weighting $\omega = \tau[f_1, f_2, ..., f_m]$, where the weighting τ is in $W_{c\neq 0}^m$ and operations $f_1, f_2, ..., f_m$ are from $(\wedge P_0)^k$. We have to prove that ω satisfies conditions (3.3) and (3.4).

From Theorem 6 we know that every τ from $W_{c\neq 0}$ can be generated by the weighting $\omega_\wedge$, which does not generate any weighting outside the weighted clone $W_\wedge$. Since by transitivity $\omega_\wedge$ generates $\tau[f_1, ..., f_m]$ for every $f_1, f_2, ..., f_m$ from $(\wedge P_0)^k$, then for each nonempty set of coordinates $I \subseteq \{1, ..., k\}$

$$\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J^k) = \sum_{\emptyset \neq J \subseteq I} \tau[f_1, ..., f_m](\wedge_J^k) \leq 0$$

and ω satisfies condition (3.3).

Now let's prove that for each nonempty set of coordinates $T \subseteq \{1, ..., k\}$ such that $T \cap \{j : \omega(\pi_j^k) < 0\} \neq \emptyset$, $\sum_{\emptyset \neq J \subseteq T} \omega(\wedge_J^k) < 0$. Suppose that there exist a set of coordinates $T' \subseteq \{1, ..., k\}$ such that for some $j \in T'$

$$\omega(\pi_j^k) = \tau[f_1, ..., f_m](\pi_j^k) < 0, \textit{ but } \sum_{\emptyset \neq J \subseteq T'} \omega(\wedge_J^k) = \sum_{\emptyset \neq J \subseteq T'} \tau[f_1, ..., f_m](\wedge_J^k) = 0.$$

Note that we can get the weight $\tau[f_1, ..., f_m](\pi_j^k)$ on π_j^k only by substitution of some set of projections $\{\pi_{s_1}^m, ..., \pi_{s_r}^m\}$ in $\tau[\pi_1^m, \pi_2^m, ..., \pi_m^m]$ to π_j^k. Thus, there exist a set of coordinates $\{s : f_s = \pi_t^k,\ where\ t \in T'\} = T'' \subseteq \{1, ..., m\}$ such that for some $s \in \{s_1, ..., s_r\}$

$$\tau(\pi_s^m) < 0,\ but \sum_{\emptyset \neq J \subseteq T''} \tau(\wedge_J^m) = 0.$$

But it is contradiction, because $\tau \in W_{c\neq 0}$. Thus, ω satisfies condition (3.4) and $W_{c\neq 0}$ is a weighted clone.

It remains to prove that $W_{c\neq 0}$ contains all other nontrivial weighted clones, except the weighted clone $W_\wedge$. Consider an arbitrary k-ary nonzero proper weighting ω which violates one of the conditions (3.3), (3.4). If ω violates condition (3.3), i.e. there exist a subset $I \subseteq \{1, ..., k\}$ such that $\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J) > 0$, then, as we know from the proof of Theorem 6, ω generates the trivial weighted clone $W_{\wedge P_0}$. If ω violates condition (3.4), then there exist some set of coordinates $I \subseteq \{1, ..., k\}$ such that $I \cap \{j : \omega(\pi_j) < 0\} \neq \emptyset$ and $\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J) = 0$. We can assume that $I = \{1, 2, ..., p\}$. Then the superposition

$$\omega' = \omega[\pi_1, ..., \pi_p, C_0, ..., C_0] = \sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J) \wedge_J + \Big(c + \sum_{\substack{J' \subseteq \{1,...,k\} \\ J' \cap \{p+1,...,k\} \neq \emptyset}} \omega(\wedge_J)\Big) C_0$$

is a proper nonzero weighting with the zero weight on the constant C_0. Due to Theorem 6, ω' generates the weighted clone $W_\wedge$. Since every proper nonzero weighting, which is not in $W_{c\neq 0}$, generates either $W_{\wedge P_0}$ or $W_\wedge$, then the weighted clone $W_{c\neq 0}$ contains all nontrivial weighted clones, except $W_\wedge$.

□

Now to describe all binary weighted clone over the clone $\wedge P_0$ we can formulate the following Theorem 8.

Theorem 8. *Every nontrivial binary weighted clone over the clone $\wedge P_0$ is either $\mathbf{BP}(W_\wedge)$ or is equal to $W(M)$ for some $M \subseteq \mathbb{Q}^2_{\geq 0}$ satisfying Property $(*)$. For every two sets M, M' satisfying $(*)$, $W(M) \subseteq W(M')$ if and only if $M \subseteq M'$, and $W(M) \subseteq \mathbf{BP}(W_\wedge)$.*

Proof. Consider nontrivial binary weighted clone W that contains an arbitrary nonzero binary weighting $\omega = -a_1\pi_1 - a_2\pi_2 + (a_1 + a_2)\wedge$ with the zero weight on the constant C_0. As we know from the proof of Lemma 5, $\omega \to \omega_\wedge$. By Theorem 6, W contains $\mathbf{BP}(W_\wedge)$ and since $W_\wedge$ is the largest nontrivial weighted clone, $W = \mathbf{BP}(W_\wedge)$ and contains all other binary weighted clones.

Now consider an arbitrary nontrivial binary weighted clone W over the clone $\wedge P_0$ that contains only binary weightings with nonzero weight on the constant C_0 (except the zero weighting). Due to Lemma 5, the set $M(W)$ satisfies Property $(*)$.

To prove the first part of the claim it is enough to verify that $W(M(W)) = W$. Consider an arbitrary nonzero binary weighting $\omega \in W$. Since $\omega(C_0) \neq 0$, then $\omega = s\omega_{a_1 a_2}$ for some nonnegative rationals s, a_1, a_2. Therefore $(a_1, a_2) \in M(W)$. Since ω is proper, then $a_1 + a_2 \geq 1$ and $\omega = s\omega_{a_1 a_2} \in W(M(W))$. On the other hand, according to Definition 29 each binary weighting $\omega \in W(M(W))$ is proper and of the form $s\omega_{a_1 a_2}$ for some $s \geq 0$ and $(a_1, a_2) \in M(W)$, where $a_1 + a_2 \geq 1$. That implies $\omega_{a_1 a_2} \in \text{Norm}(W)$. Since W is closed under nonnegative scaling, then $\omega = s\omega_{a_1 a_2} \in W$. Therefore, $W(M(W)) = W$.

For the second part of the claim note that if $M \subseteq M'$, then $W(M) \subseteq W(M')$ directly from the definition of $W(M)$. On the other hand, if there exist a point $(a_1, a_2) \in M$ such that $(a_1, a_2) \notin M'$, then $a_1 + a_2 \geq 1$ (since both M and M' contain the triangle with the vertices $(1,0), (0,0), (0,1)$). Thus, the weighting $\omega_{a_1 a_2}$ is proper and is in $W(M)$ but not in $W(M')$. Therefore $W(M) \nsubseteq W(M')$. □

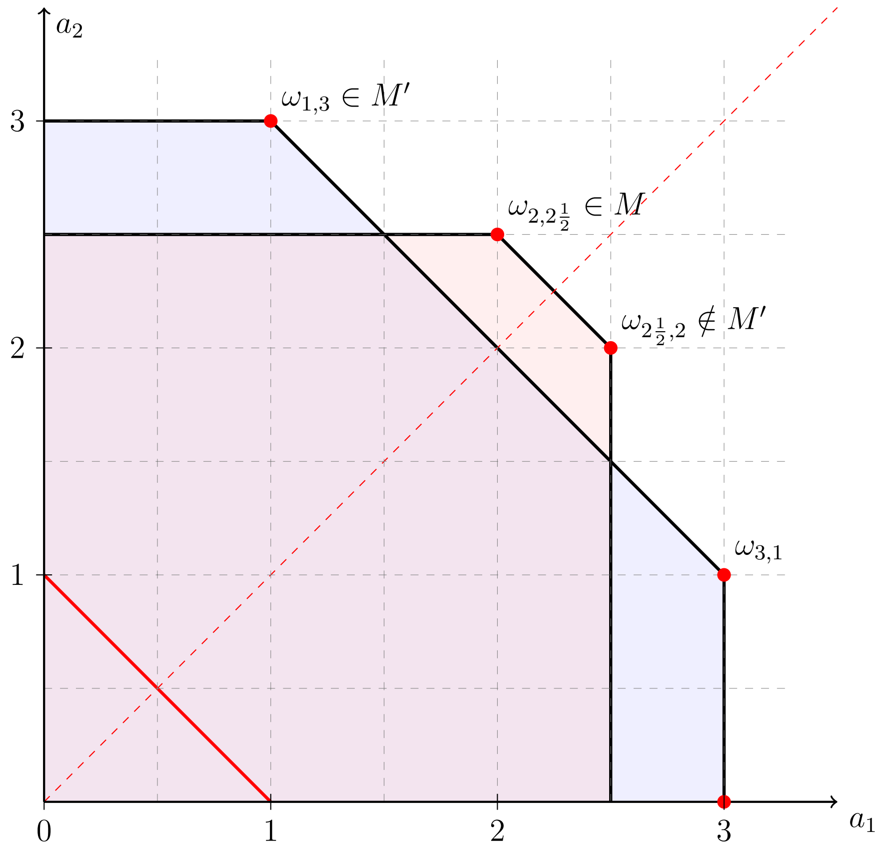

Figure 3.4: Illustration to Theorem 8.

It follows from Theorem 8 that there are binary weighted clones, which are not finitely generated. Moreover, there is continuum many binary weighted clones. A 2-generated and an infinitely generated binary weighted clones are shown in the Figure 3.5.

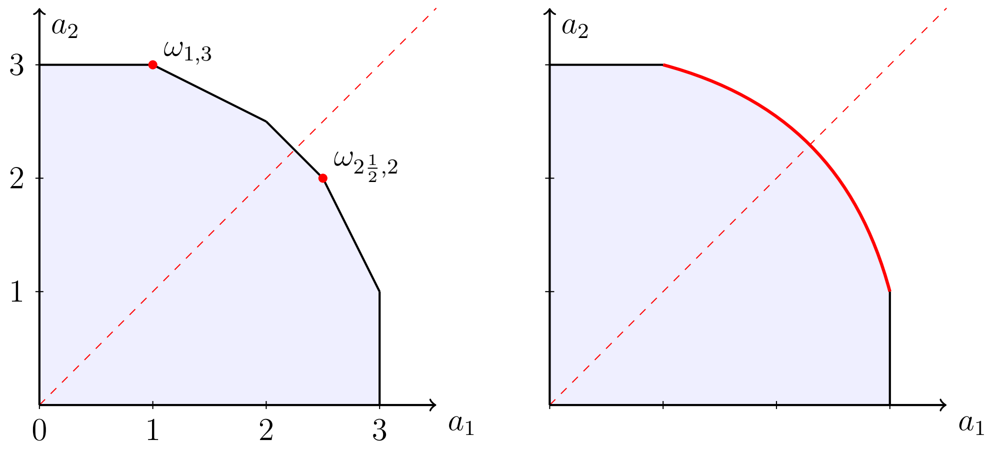

Figure 3.5: 2-generated and infinitely generated binary weighted clones.

To illustrate the lattice of binary weighted clones over the clone $\wedge P_0$ we introduce the following notation. Given a binary weighting in normed form $\omega_{a_1a_2}$ we denote the binary weighted clone generated by this weighting by $W_{a_1a_2}$. (Given a set of binary weightings in normed form $\omega_{a_1a_2}$, $\omega_{b_1b_2}$, ... we denote the binary weighted clone, generated by those weightings by $W_{a_1a_2,b_1b_2,\ldots}$.)

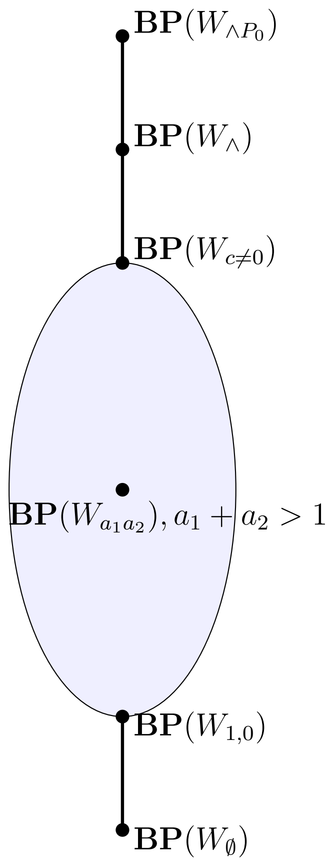

Figure 3.6: The lattice of binary weighted clones over the clone $\wedge P_0$.

Since the dual operations to $\wedge$, C_0 are $\vee$ and C_1, weighted clones over the clone $\vee P_1$=Clone($\{\vee, C_1\}$) have the similar structure.

Corollary 2. *The largest nontrivial weighted clone over the clone $\vee P_1$ is $W_\vee$, defined as follows: a k-ary weighting ω is in $W_\vee$ if and only if for each nonempty set of coordinates $I \subseteq \{1, ..., k\}$*

$$\sum_{\emptyset \neq J \subseteq I} \omega(\vee_J) \leq 0. \tag{3.5}$$

Moreover $W_\vee$ is generated by the weighting $\omega_\vee = -\pi_1 - \pi_2 + 2\vee$.

The smallest nontrivial weighted clone over the clone $\vee P_1$ is $W_{1,0}$, defined as follows: for every k

$$W_{1,0}^k = \{\omega \in W_{\vee P_1} : \text{ for every } I \subseteq \{1, ..., k\} \text{ such that } |I| > 1,\ \omega(\vee_I) = 0\}.$$

Moreover $W_{1,0}$ is generated by the unary weighting $\omega_{1,0} = -\pi_1 + C_1$.

Corollary 3. *The nontrivial weighted clone over the clone $\vee P_1$ that contains all others nontrivial weighted clones except the weighted clone $W_\vee$ is $W_{c\neq 0}$, defined as follows: a k-ary weighting ω is in $W_{c\neq 0}$ if and only if for each nonempty set of coordinates $I \subseteq \{1, ..., k\}$ and for each nonempty set of coordinates $T \subseteq \{1, ..., k\}$ such that $T \cap \{j : \omega(\pi_j^k) < 0\} \neq \emptyset$,*

$$\sum_{\emptyset \neq J \subseteq I} \omega(\vee_J^k) \leq 0, \tag{3.6}$$

$$\sum_{\emptyset \neq J \subseteq T} \omega(\vee_J^k) < 0. \tag{3.7}$$

Corollary 4. *Every nontrivial binary weighted clone over the clone $\vee P_1$ is either $\mathbf{BP}(W_\vee)$ or is equal to $W(M)$ for some $M \subseteq \mathbb{Q}^2_{\geq 0}$ satisfying Property $(*)$. For every two sets M, M' satisfying $(*)$, $W(M) \subseteq W(M')$ if and only if $M \subseteq M'$, and $W(M) \subseteq \mathbf{BP}(W_\vee)$.*

3.2.2 Binary weighted clones over the clone $\wedge P_1$

Now we will consider weighted clones over the clone $\wedge P_1$=Clone($\{\wedge, C_1\}$). The structure of weighted clones over the clone $\wedge P_1$ is very similar to the one over the clone $\wedge P_0$. The essential difference between those two structures is in the fact that in the most cases we can remove positive weight from C_1.

Lemma 7. *For every nonnegative a_1, a_2, $b_1 + b_2 > 1$ the following is true over the clone $\wedge P_1$:*

(1) $\omega_\wedge$ and $\omega_{1,0}$ are incomparable and $\omega_\wedge \not\twoheadrightarrow \omega_{a_1 a_2}$, $\omega_{1,0} \not\twoheadrightarrow \omega_{b_1 b_2}$;

(2) if at least one of the two coefficients $a_1, a_2 > 1$ then $wClone(\omega_{a_1 a_2}) = W_{\wedge P_1}$. Otherwise, $\omega_{a_1 a_2}$ and $\omega_\wedge$ are incomparable and $\omega_{a_1 a_2} \to \omega_{1,0}$.

Proof. (1) It is sufficient to note that we cannot get positive weight on $\wedge$ by any superposition of $\omega_{1,0}$ and since

$$\begin{aligned}
&\omega_\wedge[C_1, C_1] = 0,\\
&\omega_\wedge[C_1, \pi_i] = \omega_\wedge[\pi_i, C_1] = \pi_i - C_1, i = 1, 2,\\
&\omega_\wedge[C_1, \wedge] = \omega_\wedge[\wedge, C_1] = \wedge - C_1,
\end{aligned}$$

then we cannot get positive weight on C_1 by any superposition of $\omega_\wedge$.

(2) There are $4^2 = 16$ binary superpositions of $\omega_{a_1a_2}$.

The superpositions $\omega_{a_1a_2}[\pi_1, \pi_2]$, $\omega_{a_1a_2}[\pi_2, \pi_1]$, $\omega_{a_1a_2}[C_1, C_1]$, $\omega_{a_1a_2}[\wedge, \wedge]$ and $\omega_{a_1a_2}[\pi_i, \pi_i]$, $\omega_{a_1a_2}[\pi_i, \wedge]$, $\omega_{a_1a_2}[\wedge, \pi_i]$ for $i = 1, 2$ are the same as in the case of the weighting $\omega_{a_1a_2}$ over the clone $\wedge P_0$ with the only difference that instead of the constant C_0 there is the constant C_1.

The last 6 superpositions differ because of properties of the constant C_1. For $i = 1, 2$

$$\omega_{a_1a_2}[\pi_i, C_1] = -a_1\pi_i - a_2C_1 + (a_1 + a_2 - 1)\pi_i + C_1 = (a_2 - 1)\pi_i + (1 - a_2)C_1,$$
$$\omega_{a_1a_2}[C_1, \pi_i] = (a_1 - 1)\pi_i + (1 - a_1)C_1,$$
$$\omega_{a_1a_2}[\wedge, C_1] = -a_1 \wedge -a_2C_1 + (a_1 + a_2 - 1) \wedge +C_1 = (a_2 - 1) \wedge +(1 - a_2)C_0,$$
$$\omega_{a_1a_2}[C_1, \wedge] = (a_1 - 1) \wedge +(1 - a_1)C_0.$$

Thus, if neither $a_1 > 1$ nor $a_2 > 1$, then we cannot get negative weight on C_1. Therefore $\omega_{a_1a_2} \not\twoheadrightarrow \omega_\wedge$. Since $\omega_\wedge \not\twoheadrightarrow \omega_{a_1a_2}$, then $\omega_{a_1a_2}$ and $\omega_\wedge$ are incomparable.

Consider the conical combination

$$\begin{aligned}\delta &= \frac{(a_1 + a_2 - 1)}{(1 - a_1)}\omega_{a_1a_2}[C_1, \wedge] + \omega_{a_1a_2} = \\ &= (a_1 + a_2 - 1)(- \wedge +C_1) - a_1\pi_1 - a_2\pi_2 + (a_1 + a_2 - 1) \wedge +C_1 = \\ &= -a_1\pi_1 - a_2\pi_2 + (a_1 + a_2)C_1.\end{aligned}$$

Then

$$\frac{1}{a_1 + a_2}\delta[\pi_1, \pi_1] = -\pi_1 + C_1 = \omega_{1,0}.$$

Since $\omega_{a_1a_2} \to \delta \to \omega_{1,0}$, according to transitivity, $\omega_{a_1a_2} \to \omega_{1,0}$.

On the other hand, if at least one of the two coefficients $a_1, a_2 > 1$ (without loss of generality assume $a_1 > 1$), then the conical combination

$$\theta = \omega_{a_1a_2}[\pi_2, \pi_2] + \frac{1}{a_1 - 1}\omega_{a_1a_2}[C_1, \pi_1] = -\pi_2 + C_1 + \pi_1 - C_1 = \pi_1 - \pi_2$$

is a proper weighting and has positive weight on the projection π_1. Therefore by Theorem 5, $\omega_{a_1a_2}$ generates the whole $W_{\wedge P_1}$.

□

Corollary 5. *Let ω be a proper k-ary weighting ω with nonzero weight on the constant over the clone $\wedge P_1$. Assume that there exists a nonempty set of coordinates $I \subseteq \{1, ..., k\}$ such that*

$$\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J) < -\omega(C_1). \tag{3.8}$$

Then ω generates the trivial weighted clone $W_{\wedge P_1}$.

Proof. Without loss of generality we can assume that $I = \{1, ..., p\}$. Then the conical combination

$$\begin{aligned}\theta &= \frac{1}{\omega(C_1)}\omega[\pi_2, ..., \pi_2] + \frac{1}{-\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J) - \omega(C_1)}\omega[\underbrace{C_1, ..., C_1}_{p}, \pi_1, ..., \pi_1] = \\ &= -\pi_2 + C_1 + \pi_1 - C_1 = \pi_1 - \pi_2\end{aligned}$$

is a proper weighting with positive weight on the projection π_1. □

Corollary 6. *Let $\omega_{a_1a_2}$ be a normed (possibly improper) weighting over the clone $\wedge P_1$ such that $0 \le a_1, a_2 \le 1$ and let*

$$P = \{(a_1, a_2), (a_2, a_1), (a_1, 0), (0, a_1), (a_2, 0), (0, a_2), (0, 1), (1, 0), (0, 0)\}.$$

Then

1. *For each point $(a_1', a_2') \in P$ there exist binary operations f, g from $\wedge P_1$ such that $\omega_{a_1a_2}[f, g]$ is a positive multiple of $\omega_{a_1'a_2'}$.*
2. *For each binary operations f, g from $\wedge P_1$, $\omega_{a_1a_2}[f, g]$ is either the zero weighting or a positive multiple of $\omega_{a_1'a_2'}$, where $(a_1', a_2') \in P$.*

Although definitions of the objects $M(W)$ and $W(M)$ over the clone $\wedge P_0$ are the same as of the ones over the clone $\wedge P_0$, the lemmas about them differ a little.

Lemma 8. *For every nontrivial binary weighted clone W over the clone $\wedge P_1$ the set $M(W)$ is either empty or satisfies Property $(*)$.*

Proof. Consider a binary weighted clone W that does not contain nonzero weightings with nonzero weight on the constant C_1. Since for an arbitrary binary weighting $\omega = -a_1\pi_1 + a_2\pi_2 + (a_1 + a_2)\wedge$ from W

$$\frac{1}{a_1 + a_2}(\omega + \omega[\pi_2, \pi_1]) = -\pi_1 - \pi_2 + 2\wedge = \omega_\wedge,$$

and on the other hand

$$a_1\omega_\wedge[\pi_1, \wedge] + a_1\omega_\wedge[\pi_2, \wedge] = \omega,$$

then $\omega_\wedge$ and ω are equivalent. By Lemma 7 for any $b_1, b_2 \ge 0$ the weighting $\omega_\wedge$ does not generate the weighting $\omega_{b_1b_2}$. Thus, the set $M(W)$ is empty.

Now consider a binary weighted clone W that contains an arbitrary normed weighting $\omega_{a_1a_2}$. Replacing C_0 with C_1 in reasoning of Lemmas 5 proof we prove that $M(W)$ satisfies Property $(*)$ as well. □

Lemma 9. *For every set $M \subseteq \mathbb{Q}^2_{\ge 0}$ that satisfies Property (*) and is contained in the square with vertices $(1, 0), (0, 0), (0, 1), (1, 1)$, $W(M)$ is a binary weighted clone.*

Proof. The proof of this Lemma is exactly the same as for Lemma 6, since the restriction to be contained in the square with vertices $(1, 0), (0, 0), (0, 1), (1, 1)$ does not affect the properties of M. □

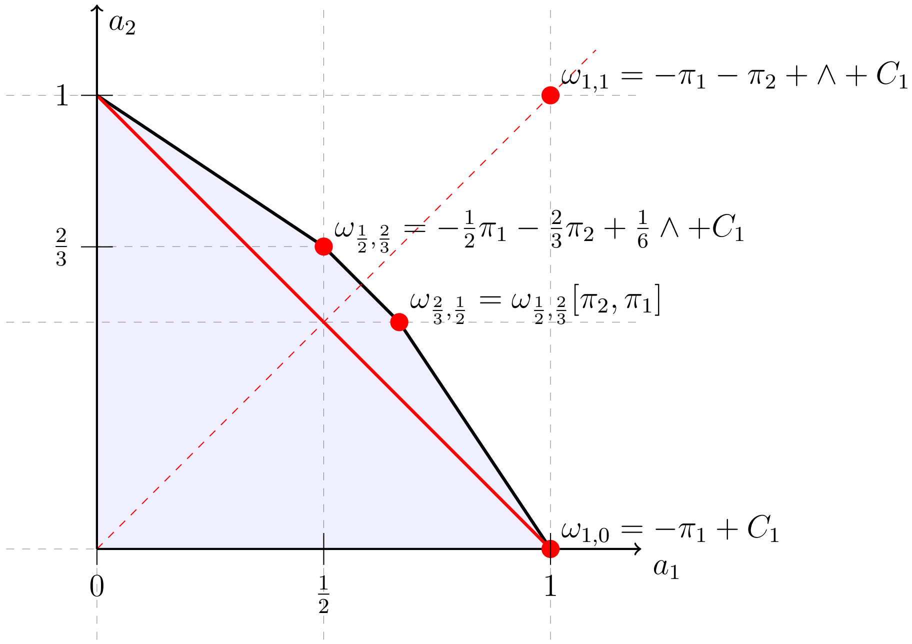

Figure 3.7: Illustration to Lemma 8.

Before we formulate the main result about the classification of all binary weighted clones over the clone $\wedge P_1$ we consider Theorem 9 concerning two atomic weighted clones.

Theorem 9. *There are two atomic weighted clones over the clone* $\wedge P_1$*:*

(1) the maximal weighted clone $W_\wedge$*, defined as follows: a* k*-ary weighting* ω *is in* $W_\wedge$ *if and only if* $\omega(C_1) = 0$ *and for each nonempty set of coordinates* $I \subseteq \{1, \ldots, k\}$

$$\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J) \leq 0. \tag{3.9}$$

Moreover, $W_\wedge$ *is generated by the weighting* $\omega_\wedge = -\pi_1 - \pi_2 + 2\wedge$*;*

(2) the weighted clone $W_{1,0}$*, defined as follows: for every* k

$$W^k_{1,0} = \{\omega \in W_{\wedge P_1} : \text{ for every } I \subseteq \{1, \ldots, k\} \text{ such that } |I| > 1,\ \omega(\wedge_I) = 0\}.$$

Moreover, $W_{1,0}$ *is generated by the unary weighting* $\omega_{1,0} = -\pi_1 + C_1$*.*

Proof. Firstly, the proof that $W_\wedge = wClone(\omega_\wedge)$ coincides with the proof of the same fact given by Jiří Vančura over the clone $\wedge = Clone(\wedge)$ (see [8]).

Secondly, the proof that $W_{1,0} = wClone(\omega_{1,0})$ coincides with the proof of the same fact in Theorem 6 if we change the constant C_0 to the constant C_1.

Finally, it is clear from Lemma 7 that $W_{1,0} \nsubseteq W_\wedge$ and $W_\wedge \nsubseteq W_{1,0}$. Thus, it remains to show that $W_{1,0}$ and $W_\wedge$ are atomic, i.e. that each nonzero weighting generates a weighted clone that contains $W_\wedge$ or $W_{1,0}$. Indeed, let τ be an arbitrary k-ary nonzero weighting. We have to consider two cases. If τ has the zero weight on the constant C_1, then $\tau \to \omega_\wedge$. Thus, τ generates the weighted clone $W_\wedge$ (see [8]). If $\tau(C_1) = c > 0$, consider the superposition:

$$\frac{1}{c}\tau[\pi_1, \ldots, \pi_1] = \frac{1}{c}\Big(c\pi_1 + cC_0\Big) = -\pi_1 + C_0.$$

That is, $wClone(\omega_{1,0}) \subseteq wClone(\tau)$. Therefore $W_{1,0}$ and $W_\wedge$ are atomic weighted clones over the clone $\wedge P_1$. □

Corollary 7. *For an arbitrary proper k-ary weighting ω with positive weight on the constant and for an arbitrary nonzero proper m-ary weighting τ with zero weight on the constant $wClone(\omega, \tau) = W_{\wedge P_1}$.*

We now describe another maximal weighted clone over the clone $\wedge P_1$.

Theorem 10. *The maximal weighted clone over the clone $\wedge P_1$ that contains all others nontrivial weighted clones except the weighted clone $W_\wedge$ is $W_{\leq -c}$, defined as follows: a k-ary weighting ω is in $W_{\leq -c}$ if and only if for each $j \in \{1, ..., k\}$ and for each nonempty set of coordinates $\emptyset \neq K \subseteq I \subseteq \{1, ..., k\}$*

$$\omega(\pi_j) \leq 0, \tag{3.10}$$

$$\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J) \leq \sum_{\emptyset \neq J \subseteq K} \omega(\wedge_J). \tag{3.11}$$

Proof. We first prove an auxiliary statement.

Claim 1. *A k-ary weighting ω is in $W_{\leq -c}$ if and only if for each binary operations $g_1, g_2, ..., g_k$ from $(\wedge P_1)^2$ a weighting $\phi = \omega[g_1, g_2, ..., g_k]$ for $i = 1, 2$ satisfies the condition*

$$-\phi(C_1) \leq \phi(\pi_i) \leq 0. \tag{3.12}$$

Proof of Claim 1. Consider an arbitrary k-ary weightings $\omega \in W_{\leq -c}$ and the superposition $\phi = \omega[g_1, g_2, ..., g_k]$, where $g_1, g_2, ..., g_k \in (\wedge P_1)^2$. There are only four binary operations containing in $(\wedge P_1)$: the two projections π_1, π_2, the meet operation $\wedge$ and the constant operation C_1. Let $T_{C_1} = \{j : g_j = C_1\}$, $T_{\pi_i} = \{j : g_j = \pi_i\}$. Note that

$$\phi(\pi_i) = \sum_{\substack{\emptyset \neq J \subseteq T_{C_1} \cup T_{\pi_i}, \\ J \cap T_{\pi_i} \neq \emptyset}} \omega(\wedge_J) = \sum_{\emptyset \neq J \subseteq T_{C_1} \cup T_{\pi_i}} \omega(\wedge_J) - \sum_{\emptyset \neq J \subseteq T_{C_1}} \omega(\wedge_J) \leq 0,$$

since ω satisfies condition (3.11) and

$$\phi(C_1) = \sum_{\emptyset \neq J \subseteq T_{C_1}} \omega(\wedge_J) + \omega(C_1).$$

Therefore

$$\phi(\pi_i) + \phi(C_1) = \sum_{\emptyset \neq J \subseteq T_{C_1} \cup T_{\pi_i}} \omega(\wedge_J) - \sum_{\emptyset \neq J \subseteq T_{C_1}} \omega(\wedge_J) + \sum_{\emptyset \neq J \subseteq T_{C_1}} \omega(\wedge_J) + \omega(C_1) =$$

$$= \sum_{\emptyset \neq J \subseteq T_{C_1} \cup T_{\pi_i}} \omega(\wedge_J) + \omega(C_1) =$$

$$= \sum_{\emptyset \neq J \subseteq T_{C_1} \cup T_{\pi_i}} \omega(\wedge_J) - \sum_{\emptyset \neq J \subseteq \{1, ..., k\}} \omega(\wedge_J) \geq 0,$$

again, since ω satisfies condition (3.11). Thus, ϕ satisfies conditions (3.12).

On the other hand, assume that proper weighting ω is not in $W_{\leq -c}$. If ω violates condition (3.10), then there exist $j \in \{1, ..., k\}$ such that $\omega(\pi_j) > 0$. Then the superposition

$$\phi = \omega[\pi_1, \pi_1, ...\pi_2, ..., \pi_1]$$

with the second projection on the j-th coordinate violates condition (3.12), since $\phi(\pi_2) > 0$. If ω violates condition (3.11), then there exist two nonempty sets $\emptyset \neq K \subseteq I \subseteq \{1, ..., k\}$, such that $\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J) > \sum_{\emptyset \neq J \subseteq K} \omega(\wedge_J)$. Without loss of generality we can assume that $K = \{1, ..., p\}$, $I = \{1, ..., p + t\}$ for some $0 < p, t$. Then superposition

$$\phi = \omega[\underbrace{C_1, ..., C_1}_{p}, \underbrace{\pi_1, ..., \pi_1}_{t}, g_{(p+t+1)}, ..., g_m]$$

violates condition (3.12), since

$$\phi(\pi_1) = \sum_{\emptyset \neq J \subseteq \{1,...,(p+t)\}} \omega(\wedge_J) - \sum_{\emptyset \neq J \subseteq \{1,...,p\}} \omega(\wedge_J) > 0.$$

Now we are ready to prove Theorem 10. We first show that $W_{\leq -c}$ is a weighted clone. It is obvious that $W_{\leq -c}$ is closed under conical combinations of weightings. Thus, we only have to prove that $W_{\leq -c}$ is closed under proper superposition. Consider a proper nonzero m-ary weighting $\tau = \omega[f_1, f_2, ..., f_k]$, where the nonzero weighting ω is in $W^k_{\leq -c}$ and operations $f_1, f_2, ..., f_k$ are from $(\wedge P_1)^m$. We have to verify conditions (3.7) and (3.8) for τ.

According to the Claim above, it is sufficient to show that for each binary operations $g_1, g_2, ..., g_m$ from $(\wedge P_1)^2$ the weighting $\phi = \omega[f_1, f_2, ..., f_k][g_1, ..., g_m]$ satisfies condition (3.12). But the superposition

$$\phi = \omega[f_1, f_2, ..., f_k][g_1, ..., g_m] = \omega[f_1(g_1, ..., g_m), ..., f_k(g_1, ..., g_m)],$$

where $f_1(g_1, ..., g_m), ..., f_k(g_1, ..., g_m)$ are binary operations. Thus, ϕ satisfies condition (3.12), since $\omega \in W_{\leq -c}$.

It remains to prove that $W_{\leq -c}$ contains all other nontrivial weighted clones, except the weighted clone $W_\wedge$. Consider an arbitrary k-ary nonzero proper weighting ω that violates conditions (3.10), (3.11). If ω violates condition (3.10), then by Theorem 5, $wClone(\omega) = W_{\wedge P_1}$. Assume that ω violates condition (3.11). Then there exist two nonempty sets $\emptyset \neq K \subseteq I \subseteq \{1, ..., k\}$, such that $\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J) > \sum_{\emptyset \neq J \subseteq K} \omega(\wedge_J)$. Again, without loss of generality assume that $K = \{1, ..., p\}$, $I = \{1, ..., p + t\}$. First note that conditions (3.10), (3.11) imply $\omega(C_1) > 0$. If $\omega(C_1) = 0$, then $\omega \to \omega_\wedge$ and $W_\wedge \subseteq wClone(\omega)$. If $\omega(C_1) > 0$, then the conical combination

$$\theta = \omega[\underbrace{C_1, ..., C_1}_{p}, \underbrace{\pi_1, ..., \pi_1}_{t}, \pi_{(p+t+1)}, ..., \pi_m] + \frac{\sum_{\emptyset \neq J \subseteq \{1,...,p\}} \omega(\wedge_J)}{\omega(C_1)} \omega[\pi_2, ..., \pi_2]$$

is proper weighting with positive weight on the projection π_1 and therefore generates $W_{\wedge P_0}$.

□

Now to describe all binary weighted clone over the clone $\wedge P_1$ we can formulate the following Theorem 11.

Theorem 11. *Every nontrivial binary weighted clone over the clone* $\wedge P_1$ *is either* $\mathbf{BP}(W_\wedge)$ *or is equal to* $W(M)$ *for some* M *that satisfies Property (∗) and is contained in the square with vertices* $(1,0),(0,0),(0,1),(1,1)$*. For every two such sets* M*,* M' *the binary weighted clone* $\mathbf{BP}(W_\wedge)$ *is incomparable with* $W(M)$ *and* $W(M) \subseteq W(M')$ *if and only if* $M \subseteq M'$*.*

Proof. Consider an arbitrary binary weighted clone W. From Corollary 7 we know, that if W contains both a nonzero weighting with the zero weight on the constant C_1 and a nonzero weighting with positive weight on the constant, then W is the binary part of the trivial weighted clone $\wedge P_1$. Indeed, in this case W contains the weighting $\omega_{a_1 a_2}$ with at least one of the coefficients a_1, a_2 is greater than 1, which due to Lemma 7 generates all the weightings. If W does not contain a nonzero weighting with positive weight on the constant, then, by the proof of Lemma 8, W is the binary part of $W_\wedge$. Therefore, it is sufficient to consider an arbitrary binary weighted clone W over the clone $\wedge P_1$ that contains only binary weightings with positive weight on the constant C_1 (except the zero weighting) and which, being normed, have coefficients $a_1, a_2 \leq 1$. The rest of the proof is the same as in Theorem 8. □

It follows from Theorem 11 that there are binary weighted clones, which are not finitely generated. Moreover, there is continuum many binary weighted clones.

To illustrate the lattice of binary weighted clones over the clone $\wedge P_1$ we use the notation of Section 3.2.1.

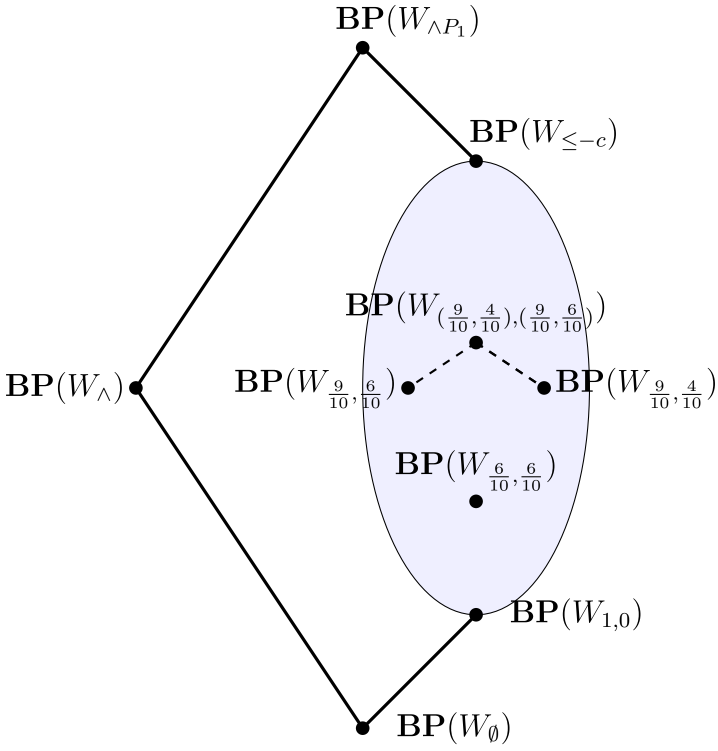

Figure 3.8: wedgeThe lattice of binary weighted clones over the clone $\wedge P_1$.

The dual operations to $\wedge$ and C_1 are $\vee$ and C_0. Hence the weighted clones over the clone $\vee P_0$=Clone($\{\vee, C_0\}$) have the same structure as $\wedge P_1$.

Corollary 8. *There are two atomic weighted clones over the clone $\vee P_0$:*

(1) the maximal weighted clone $W_\vee$, defined as follows: a k-ary weighting ω is in $W_\vee$ if and only if $\omega(C_0) = 0$ and for each nonempty set of coordinates $I \subseteq \{1, ..., k\}$

$$\sum_{\emptyset \neq J \subseteq I} \omega(\vee_J) \leq 0. \tag{3.13}$$

Moreover, $W_\vee$ is generated by the weighting $\omega_\vee = -\pi_1 - \pi_2 + 2\vee$;

(2) the weighted clone $W_{1,0}$, defined as follows: for every k

$$W^k_{1,0} = \{\omega \in W_{\vee P_0} : \textit{ for every } I \subseteq \{1, ..., k\} \textit{ such that } |I| > 1,\ \omega(\vee_I) = 0\}.$$

Moreover, $W_{1,0}$ is generated by the unary weighting $\omega_{1,0} = -\pi_1 + C_0$.

Corollary 9. *The maximal weighted clone over the clone $\vee P_0$ that contains all others nontrivial weighted clones except the weighted clone $W_\vee$ is $W_{\leq -c}$, defined as follows: a k-ary weighting ω is in $W_{\leq -c}$ if and only if for each $j \in \{1, ..., k\}$ and for each nonempty set of coordinates $\emptyset \neq K \subseteq I \subseteq \{1, ..., k\}$*

$$\omega(\pi_j) \leq 0, \tag{3.14}$$

$$\sum_{\emptyset \neq J \subseteq I} \omega(\vee_J) \leq \sum_{\emptyset \neq J \subseteq K} \omega(\vee_J). \tag{3.15}$$

Corollary 10. *Every nontrivial binary weighted clone over the clone $\vee P_0$ is either* $\mathbf{BP}(W_\vee)$ *or is equal to $W(M)$ for some M that satisfies Property ($*$) and is contained in the square with vertices* $(1, 0), (0, 0), (0, 1), (1, 1)$. *For every two such sets M, M' the binary weighted clone* $\mathbf{BP}(W_\vee)$ *is incomparable with $W(M)$ and $W(M) \subseteq W(M')$ if and only if $M \subseteq M'$.*

3.3 Binary weighted clones over the clone $\wedge P_{01}$

We now can combine all previous results to describe the structure of weighted clones over the clone $\wedge P_{01}$ = Clone($\{\wedge, C_0, C_1\}$). Before we get to description we need to redefine several objects, because in the case over the clone with two constants, we cannot avoid parameter t in set of indices and have to work with three-dimensional space instead of a plane.

To a binary (possibly improper) weighting $\omega^t_{a_1 a_2}$ we assign a point (a_1, a_2, t) in three-dimensional space $\mathbb{Q}^3_{\geq 0}$, where

$$\mathbb{Q}^3_{\geq 0} = \{(a, b, t) \in \mathbb{Q}^3 : a, b, t \geq 0\}.$$

Thus, for example, a proper weighting $\omega^{\frac{1}{4}}_{2,3} = -2\pi_1 - 3\pi_2 + 4\wedge + \frac{3}{4}C_0 + \frac{1}{4}C_1$ corresponds to the point $(2, 3, \frac{1}{4})$, and improper weightings $\omega^0_{0,0} = -\wedge + C_0$, $\omega^1_{0,0} = -\wedge + C_1$ correspond to points $(0, 0, 0)$ and $(0, 0, 1)$ respectively.

We will also use the notation:

$$\mathbb{Q}^{t=0}_{\geq 0} = \{(a, b, 0) \in \mathbb{Q}^3 : a, b \geq 0\}.$$

$$\mathbb{Q}^{t=1}_{\geq 0} = \{(a, b, 1) \in \mathbb{Q}^3 : a, b \geq 0\}.$$

Definition 30. We say that a subset M of $\mathbb{Q}^3_{\geq 0}$ *satisfies Property (**)* if:

(1) M is convex, i.e., for every $\mathbf{x_1}, \mathbf{x_2} \in M$ and every $t \in [0,1]$ we have $t\mathbf{x_1} + (1-t)\mathbf{x_2} \in M$;

(2) For every t M is symmetric with respect to the line $x = y$, i.e. $(x,y,t) \in M \Leftrightarrow (y,x,t) \in M$;

(3) For each t if $(x,y,t) \in M$, then M contains the points $(0,1,t)$ and $(1,0,t)$. Moreover, if $0 < t < 1$, then $(x,y,0)$,$(x,y,1) \in M$;

(4) if M contains a point (x,y,t), then $(x,0,t) \in M$ and $(0,y,t) \in M$.

For the clone $\wedge P_{0,1}$ we define two objects, $M(W)$ and $W(M)$ as follows:

Definition 31. Given a binary weighted clone W over a clone C we denote by $M(W)$ the set of points in $\mathbb{Q}^3_{\geq 0}$

$$M(W) := \{(b_1, b_2, t) : \omega^t_{b_1 b_2} \textit{ is } (\textit{possibly improper}) \textit{ weighting}$$
$$\textit{such that } \omega^t_{b_1 b_2} \leftarrow \omega \in W\}.$$

Definition 32. Given a set M of points in $\mathbb{Q}^3_{\geq 0}$ we denote by $W(M)$ the set of proper binary weightings

$$W(M) := \{s\omega^t_{a_1 a_2} : (a_1, a_2, t) \in M, s \geq 0, a_1 + a_2 \geq 1\} \cup \{\textit{the zero weightings}\}.$$

We are ready now to work with binary weightings over the clone $\wedge P_{01}$. At first, instead of one big lemma we will prove several little ones to emphasize their results in construction of lattice of binary weighted clones.

Now note that since binary part of the clone $\wedge P_{01}$ contains two projections and three operations $\wedge, C_0, C_1$, then for each binary weighting there are $n^k = 25$ binary superpositions.

Lemma 10. *For every $0 \leq a_1, a_2 \leq 1$, $0 < t < 1$ the weighting $\omega^t_{a_1 a_2}$ generates both weightings $\omega^0_{a_1 a_2}$ and $\omega^1_{a_1 a_2}$ and does not generate $\omega_\wedge$ over the clone $\wedge P_{01}$.*

Proof. Consider two superpositions:

$$\omega^t_{a_1 a_2}[C_0, C_0] = -a_1 C_0 - a_2 C_0 + (a_1 + a_2 - 1)C_0 + (1-t)C_0 + tC_1 =$$
$$= -tC_0 + tC_1,$$
$$\omega^t_{a_1 a_2}[C_1, C_1] = (1-t)C_0 - (1-t)C_1.$$

Therefore, in the weighting $\omega^t_{a_1 a_2}$ we can remove the positive weight from the constant C_0, as well as from the constant C_1. Indeed,

$$\omega^0_{a_1 a_2} = \omega^t_{a_1 a_2} + \frac{t}{1-t}\omega^t_{a_1 a_2}[C_1, C_1],$$
$$\omega^1_{a_1 a_2} = \omega^t_{a_1 a_2} + \frac{1-t}{t}\omega^t_{a_1 a_2}[C_0, C_0].$$

To show that the weighting $\omega^t_{a_1a_2}$ does not generate the weighting $\omega_\wedge$ we have to consider all other possible superpositions of $\omega^t_{a_1a_2}$. There are the trivial superposition $\omega^t_{a_1a_2}[\pi_1,\pi_2]$ and the superposition $\omega^t_{a_1a_2}[\pi_2,\pi_1]$ which switches the weights on projections, there are two superpositions

$$\begin{aligned}
&\omega^t_{a_1a_2}[C_0,C_1] = -a_1C_0 - a_2C_1 + (a_1+a_2-1)C_0 + (1-t)C_0 + tC_1 = \\
&= (a_2-t)C_0 + (t-a_2)C_1, \\
&\omega^t_{a_1a_2}[C_1,C_0] = (a_1-t)C_0 + (t-a_1)C_1,
\end{aligned}$$

which are still the weightings $\theta = -C_0 + C_1$ or $\eta = C_0 - C_1$, multiplied by some nonnegative constant, there are ten superpositions, $i = 1, 2$

$$\begin{aligned}
&\omega^t_{a_1a_2}[\pi_i,\pi_i] = -\pi_i + (1-t)C_0 + tC_0, \\
&\omega^t_{a_1a_2}[\pi_i,C_0] = -a_1\pi_i - a_2C_0 + (a_1+a_2-1)C_0 + (1-t)C_0 + tC_1 = \\
&= -a_1\pi_i + (a_1-t)C_0 + tC_1, \\
&\omega^t_{a_1a_2}[C_0,\pi_i] = -a_2\pi_i + (a_2-t)C_0 + tC_1, \\
&\omega^t_{a_1a_2}[\pi_i,C_1] = -a_1\pi_i - a_2C_1 + (a_1+a_2-1)\pi_i + (1-t)C_0 + tC_1 = \\
&= -(1-a_2)\pi_i + (1-t)C_0 + (t-a_2)C_1, \\
&\omega^t_{a_1a_2}[C_1,\pi_i] = -(1-a_1)\pi_i + (1-t)C_0 + (t-a_1)C_1,
\end{aligned}$$

which have the zero weights on $\wedge$, there are five superpositions, $i = 1, 2$

$$\begin{aligned}
&\omega^t_{a_1a_2}[\wedge,\wedge] = -\wedge + (1-t)C_0 + tC_1, \\
&\omega^t_{a_1a_2}[\pi_i,\wedge] = -a_1\pi_i + (a_1-1)\wedge + (1-t)C_0 + tC_1, \\
&\omega^t_{a_1a_2}[\wedge,\pi_i] = -a_2\pi_i + (a_1-1)\wedge + (1-t)C_0 + tC_1,
\end{aligned}$$

which have non-positive weights on $\wedge$ and positive weights on the constants and, finally, there are four superpositions

$$\begin{aligned}
&\omega^t_{a_1a_2}[C_0,\wedge] = -a_2\wedge + (a_2-t)C_0 + tC_1, \\
&\omega^t_{a_1a_2}[\wedge,C_0] = -a_1\wedge + (a_1-t)C_0 + tC_1, \\
&\omega^t_{a_1a_2}[C_1,\wedge] = -(1-a_1)\wedge + (1-t)C_0 + (t-a_1)C_1, \\
&\omega^t_{a_1a_2}[\wedge,C_1] = -(1-a_2)\wedge + (1-t)C_0 + (t-a_2)C_1,
\end{aligned}$$

which have non-positive weights on $\wedge$ and (possibly) non-positive weight on the one of the constants. As we can see, by any superposition we cannot get positive weight on $\wedge$ (since $a_1, a_2 \leq 1$) and cannot get negative weights on the both constants C_0, C_1 at the same time. We only have the following seven combinations of weights on constants: $(1-t)C_0 + tC_1$, $-C_0 + C_1$, $C_0 - C_1$, $(a_1-t)C_0 + tC_1$, $(a_2-t)C_0 + tC_1$, $(1-t)C_0 + (t-a_1)C_1$ and $(1-t)C_0 + (t-a_2)C_1$. For $\omega^t_{a_1a_2}$ we have $\omega^t_{a_1a_2}(C_0) + \omega^t_{a_1a_2}(C_1) = 1$ and for every binary operation f, g over the clone $\wedge P_{01}$ we have $\omega^t_{a_1a_2}[f,g](C_0) + \omega^t_{a_1a_2}[f,g](C_1) \geq 0$. It implies that for any conical combination of superpositions of $\omega^t_{a_1a_2}$, call it ω, which contains $s\omega^t_{a_1a_2}[\pi_1,\pi_2]$, $s > 0$ (because we need $\omega(\wedge) > 0$), we cannot get $\omega(C_0) = \omega(C_1) = 0$, which means that by any conical combination of superpositions of $\omega^t_{a_1a_2}$ we cannot get a proper binary weighting with zero weights on the both constants and positive weight on $\wedge$. Therefore $\omega^t_{a_1a_2} \nrightarrow \omega_\wedge$. □

The immediate corollaries of the proof of Lemma 10 are the following:

Corollary 11. *For every* $0 < t < 1$, $0 \le s \le 1$ $\omega^t_{a_1 a_2} \to \omega^s_{a_1 a_2}$.

Corollary 12. *For every* $0 < t < 1$ $wClone(\omega^t_{a_1a_2}) = wClone(\omega^0_{a_1a_2}, \omega^1_{a_1a_2})$.

Proof. It will be sufficient to prove that there $\{\omega^0_{a_1a_2}, \omega^1_{a_1a_2}\} \to \omega^t_{a_1a_2}$. Indeed,

$$\omega^{\frac{1}{2}}_{a_1a_2} = \frac{1}{2}\omega^0_{a_1a_2} + \frac{1}{2}\omega^1_{a_1a_2},$$

and since by Corollary 11 $\omega^{\frac{1}{2}}_{a_1a_2} \to \omega^t_{a_1a_2}$, we are done. □

Corollary 13. *For every* $0 < t < 1$, $\omega^t_{a_1a_2} \to \omega^t_{1,0}$ *and* $\omega^t_{1,0} \to \omega^0_{1,0}, \omega^1_{1,0}$.

Corollary 14. *Let* $\omega^t_{a_1a_2}$ *be a normed (possibly improper) weighting over the clone* $\wedge P_{01}$ *such that* $0 < t < 1$ *and let*

$$P = \{(a_1, a_2, t), (a_2, a_1, t), (a_1, 0, t), (0, a_1, t), (a_2, 0, t), (0, a_2, t),$$
$$(1, 0, t), (0, 1, t), (0, 0, t)\}.$$

Then

1. *For each point* $(a_1', a_2', t') \in P$ *there exist binary operations* f, g *from* $\wedge P_{01}$ *such that* $\omega^t_{a_1a_2}[f, g]$ *is a positive multiple of* $\omega^{t'}_{a_1'a_2'}$.
2. *For each binary operations* f, g *from* $\wedge P_{01}$, $\omega^t_{a_1a_2}[f, g]$ *is either the zero weighting, or a positive multiple of the weightings* $\theta = -C_0 + C_1$, $\eta = C_0 - C_1$, *or a positive multiple of* $\omega^{t'}_{a_1'a_2'}$, *where* $(a_1', a_2', t') \in P$.

Lemma 11. *If* $0 < t \le 1$ *and at least one of two coefficients* a_1, a_2 *is greater than 1, then* $\omega^t_{a_1a_2}$ *generates the trivial weighted clone* $W_{\wedge P_{01}}$.

Proof. Assume that $a_1 > 1$. Consider conical combination

$$\omega^t_{a_1a_2} + \frac{(1-t)}{t}\omega[C_0, C_0] = \omega^1_{a_1a_2}.$$

The claim now follows from Lemma 7. □

Corollary 15. *Let* ω *be a proper* k*-ary weighting over the clone* $\wedge P_{01}$. *Assume that* $\omega(C_1) \ne 0$ *and there exists a nonempty set of coordinates* $I \subseteq \{1, ..., k\}$ *such that*

$$\sum_{\emptyset \ne J \subseteq I} \omega(\wedge_I) < -(\omega(C_0) + \omega(C_1)).$$

Then ω *generates the trivial weighted clone* $W_{\wedge P_{01}}$

Proof. The claim follows from Lemma 11 and Corollary 5. □

The proof of the following lemma is based mostly on Lemmas 4, 7 in sections 3.1 and 3.2.

Lemma 12. *For every nonnegative rationals* $a_1, a_2, b_1, b_2, c_1, c_2, t$ *such that* $a_1 + a_2 > 1$, $b_1, b_2, c_1, c_2 \le 1$, $b_1 + b_2 > 1$, $c_1 + c_2 > 1$, $0 \le t \le 1$ *the following is true over the clone* $\wedge P_{01}$:

(1) $\omega_\wedge \to \omega^0_{a_1a_2}$, $\omega^0_{a_1a_2} \to \omega^0_{1,0}$, $\omega^1_{a_1a_2} \to \omega^1_{1,0}$;

(2) *if* $t \neq 0$, *then* $\omega^0_{a_1a_2} \not\twoheadrightarrow \omega^t_{b_1b_2}$, $\omega^0_{a_1a_2} \not\twoheadrightarrow \omega_\wedge$, $\omega^0_{1,0} \not\twoheadrightarrow \omega^0_{a_1a_2}$;

(3) *if* $t \neq 1$, *then* $\omega^1_{c_1c_2} \not\twoheadrightarrow \omega^t_{b_1b_2}$, $\omega^1_{c_1c_2} \not\twoheadrightarrow \omega_\wedge$, $\omega^1_{1,0} \not\twoheadrightarrow \omega^0_{b_1b_2}$;

(4) *if* $t \neq 0$, *then* $\omega_\wedge \not\twoheadrightarrow \omega^t_{a_1a_2}$.

Proof. We do not need to prove the relation $\to$, because we did it in Lemmas 4, 7. To prove the other items we have to consider additional superpositions for each weighting and show that even by them we will not get new weightings. Note that the relations $\omega^0_{1,0} \not\twoheadrightarrow \omega^0_{a_1a_2}$, $\omega^1_{1,0} \not\twoheadrightarrow \omega^0_{b_1b_2}$ are trivial.

(2) There are nine new superpositions of $\omega^0_{a_1a_2}$ with constant C_1. There are three superpositions

$$\omega^0_{a_1a_2}[C_1, C_1] = -C_1 + C_0,$$
$$\omega^0_{a_1a_2}[C_0, C_1] = -a_1C_0 - a_2C_1 + (a_1 + a_2 - 1)C_0 + C_0 = -a_2C_1 + a_2C_0,$$
$$\omega^0_{a_1a_2}[C_1, C_0] = -a_1C_1 + a_1C_0,$$

which are the weighting $\eta = C_0 - C_1$, multiplied by some nonnegative constant, and there are six superpositions, $i = 1, 2$

$$\omega^0_{a_1a_2}[C_1, \pi_i] = -a_1C_1 - a_2\pi_i + (a_1 + a_2 - 1)\pi_i + C_0 = (a_1 - 1)\pi_i - a_1C_1 + C_0,$$
$$\omega^0_{a_1a_2}[\pi_i, C_1] = (a_2 - 1)\pi_i - a_2C_1 + C_0,$$
$$\omega^0_{a_1a_2}[C_1, \wedge] = -a_1C_1 - a_2 \wedge +(a_1 + a_2 - 1) \wedge +C_0 = (a_1 - 1) \wedge -a_1C_1 + C_0$$
$$\omega^0_{a_1a_2}[\wedge, C_1] = (a_2 - 1) \wedge -a_2C_1 + C_0.$$

Thus, we still cannot get rid of the positive weight on the constant C_0 and therefore $\omega^0_{a_1a_2} \not\twoheadrightarrow \omega_\wedge$, and cannot get positive weight on the constant C_1, therefore $\omega^0_{a_1a_2} \not\twoheadrightarrow \omega^t_{a_1a_2}$.

(3) There are nine new superpositions of $\omega^1_{c_1c_2}$ with constant C_0. There are three superpositions

$$\omega^1_{c_1c_2}[C_0, C_0] = -C_0 + C_1,$$
$$\omega^1_{c_1c_2}[C_0, C_1] = -c_1C_0 - c_2C_1 + (c_1 + c_2 - 1)C_0 + C_1 =$$
$$-(1 - c_2)C_0 + (1 - c_2)C_1,$$
$$\omega^1_{c_1c_2}[C_1, C_0] = -(1 - c_1)C_0 + (1 - c_1)C_1,$$

which are the weighting $\eta = C_1 - C_0$, multiplied by some nonnegative constant (since $0 \leq c_1, c_2 \leq 1$), and there are six superpositions, $i = 1, 2$

$$\omega^t_{c_1c_2}[C_0, \pi_i] = -c_1C_0 - c_2\pi_i + (c_1 + c_2 - 1)C_0 + C_1 = -c_2\pi_i - (1 - c_2)C_0 + C_1$$
$$\omega^t_{c_1c_2}[\pi_i, C_0] = -c_1\pi_i - (1 - c_1)C_0 + C_1$$
$$\omega^t_{c_1c_2}[C_0, \wedge] = -c_1C_0 - b_2 \wedge +(c_1 + c_2 - 1)C_0 + C_1 = -c_2 \wedge -(1 - c_2)C_0 + C_1$$
$$\omega^t_{c_1c_2} = -c_1 \wedge -(1 - c_1)C_0 + C_1.$$

As we can see, we cannot remove the positive weight from C_1 and therefore $\omega^1_{c_1c_2} \not\twoheadrightarrow \omega_\wedge$, and cannot get positive weight on C_0, therefore $\omega^1_{c_1c_2} \not\twoheadrightarrow \omega^t_{b_1b_2}$.

(4) There are only two new superpositions of $\omega_\wedge$ both with the constants C_0, C_1:

$$\omega_\wedge[C_0, C_1] = \omega_\wedge[C_1, C_0] = C_0 - C_1.$$

Thus, we still cannot get positive weight on C_1 and $\omega_\wedge \nrightarrow \omega^t_{a_1a_2}$.

□

Corollary 16. *For every proper binary weighting $\omega^t_{b_1b_2}$ over the clone $\wedge P_{01}$ with $t \neq 0$ the following is true:*

(1) $wClone(\omega_\wedge, \omega^t_{b_1b_2}) = W_{\wedge P_{01}}$;

(2) for every $\omega^0_{a_1a_2}$ such that at least one of the coefficients is greater than 1, $wClone(\omega^0_{a_1a_2}, \omega^t_{b_1b_2}) = W_{\wedge P_{01}}$;

Proof. (1) It is enough to note that for $i = 1, 2$ the coefficients $(b_i + 1) > 1$.

(2) Assume that $a_1 > 1$. Then the conical combination

$$\begin{aligned}
&\omega^{t'}_{b'_1b'_2} = \frac{1}{a_1}\omega^0_{a_1a_2} + \frac{a_1-1}{a_1}\omega^t_{b_1b_2} = \\
&= -(1 + b_1\frac{(a_1-1)}{a_1})\pi_1 - (\frac{a_2 + b_2(a_1-1)}{a_1})\pi_2 + \\
&+ (\frac{a_2 + b_1(a_1-1) + b_2(a_1-1)}{a_1}) \wedge + \frac{a_1(1-t)+t}{a_1}C_0 + \frac{t(a_1-1)}{a_1}C_1
\end{aligned}$$

is proper normed weighting with $t' \neq 0$ and coefficients b'_1, b'_2, where $b'_1 = (1 + b_1\frac{(a_1-1)}{a_1}) > 1$. By Lemma 11 $\omega^{t'}_{b'_1b'_2}$ generates $W_{\wedge P_{01}}$.

□

The next two corollaries follow from the previous Lemma 12 and Lemmas 4 and 7.

Corollary 17. *Let $\omega^0_{a_1a_2}$ be a normed (possibly improper) weighting over the clone $\wedge P_{01}$ and let*

$$P = \{(a_1, a_2, 0), (a_2, a_1, 0), (a_1, 0, 0), (0, a_1, 0), (a_2, 0, 0), (0, a_2, 0), \\ (1, 0, 0), (0, 1, 0), (0, 0, 0)\}.$$

Then

1. *For each point $(a'_1, a'_2, 0) \in P$ there exist binary operations f, g from $\wedge P_{01}$ such that $\omega^0_{a_1a_2}[f, g]$ is a positive multiple of $\omega^0_{a'_1a'_2}$.*
2. *For each binary operations f, g from $\wedge P_{01}$, $\omega^0_{a_1a_2}[f, g]$ is either the zero weighting or a positive multiple of $\omega^0_{a'_1a'_2}$, where $(a'_1, a'_2, 0) \in P$.*

Corollary 18. *Let $\omega^1_{a_1a_2}$ be a normed (possibly improper) weighting over the clone $\wedge P_{01}$ and let*

$$P = \{(a_1, a_2, 1), (a_2, a_1, 1), (a_1, 0, 1), (0, a_1, 1), (a_2, 0, 1), (0, a_2, 1), \\ (1, 0, 1), (0, 1, 1), (0, 0, 1)\}.$$

Then

1. *For each point* $(a_1', a_2', 1) \in P$ *there exist binary operations* f, g *from* $\wedge P_{01}$ *such that* $\omega^1_{a_1a_2}[f,g]$ *is a positive multiple of* $\omega^1_{a_1'a_2'}$.
2. *For each binary operations* f, g *from* $\wedge P_{01}$, $\omega^1_{a_1a_2}[f,g]$ *is either the zero weighting or a positive multiple of* $\omega^1_{a_1'a_2'}$, *where* $(a_1', a_2', 1) \in P$.

Now we are ready to prove the dual lemmas about the subsets of points $M(W)$ and binary weighted clones $W(M)$. Since from Lemma 12 we know, that for $t \neq 0$ $\omega^0_{a_1a_2} \not\twoheadrightarrow \omega^t_{a_1a_2}$, $\omega_\wedge \not\twoheadrightarrow \omega^t_{a_1a_2}$, and for $0 \leq b_1, b_2 \leq 1$, $t \neq 1$ $\omega^1_{b_1b_2} \not\twoheadrightarrow \omega^t_{b_1b_2}$, $\omega^1_{b_1b_2} \not\twoheadrightarrow \omega_\wedge$ we can split the proof of Lemma 13 into three cases: when $M(W)$ lies in $\mathbb{Q}^{t=0}_{\geq 0}$, when $M(W)$ lies in $\mathbb{Q}^{t=1}_{\geq 0}$, and otherwise.

Lemma 13. *For every nontrivial binary weighted clone* W *over the clone* $\wedge P_{01}$ *the set* $M(W)$ *satisfies Property* $(**)$.

Proof. (1) Assume that the binary weighted clone W does not contain nonzero weightings with nonzero weight on the constant C_1. It means, that $M(W) \subseteq \mathbb{Q}^{t=0}_{\geq 0}$. Due to Corollary 17 in that case we are in conditions of Lemma 5, and, as we have seen, in both cases (W does or does not contain a nonzero weighting with zero weight on the constant C_0) $M(W)$ satisfies Property $(*)$. Due to Definition 30, $M(W)$ satisfies Property $(**)$ as well.

(2) Now assume that the binary weighted clone W contains only weightings of the form $\omega^1_{b_1b_2}$ with $0 \leq b_1, b_2 \leq 1$. It means, that $M(W) \subseteq \mathbb{Q}^{t=1}_{\geq 0}$. Due to Corollary 18, we can use the same reasoning as in proof of Lemma 8 and will get that $M(W)$ satisfies Property $(*)$. Thus, $M(W)$ satisfies Property $(**)$.

(3) Finally, otherwise, if $M(W) \not\subseteq \mathbb{Q}^{t=0}_{\geq 0}$ and $M(W) \not\subseteq \mathbb{Q}^{t=1}_{\geq 0}$, then $M(W)$ contains some point (a_1, a_2, t) with $0 < t < 1$, which corresponds to the weighting $\omega^t_{a_1a_2}$. Indeed, if we assume, that $M(W) \subseteq \mathbb{Q}^{t=0}_{\geq 0} \cup \mathbb{Q}^{t=1}_{\geq 0}$, then according to Definition 31 for every points $(c_1, c_2, 0)$ and $(b_1, b_2, 1)$ there exist proper binary weightings $\omega^r_{c_1'c_2'}$ and $\omega^s_{b_1'b_2'} \in W$ that generate the weightings $\omega^0_{c_1c_2}$ and $\omega^1_{b_1b_2}$. Even if $r = 0, s = 1$, since W is closed under nonnegative scaling and sum of weightings, then the weighting $\omega^t_{a_1a_2} = \frac{1}{2}\omega^r_{c_1c_2} + \frac{1}{2}\omega^s_{b_1b_2}$ with $t = \frac{1}{2}$ is in W and therefore the point $(a_1, a_2, t) \in M(W)$.

If at least one of the coefficients $a_1, a_2 > 1$, then $\omega^t_{a_1a_2}$ generate all weightings over the clone $\wedge P_{01}$, i.e. $wClone(\omega^t_{a_1a_2}) = W_{\wedge P_{01}}$. Hence the set $M(W)$ is equal to the whole $\mathbb{Q}^3_{\geq 0}$ and satisfies $(**)$. Otherwise, we have to check that $M(W)$ satisfies all conditions $(1), (2), (3), (4)$ in the definition of $(**)$.

We first prove that $M(W)$ is convex. Again, as we did it in the proof of Lemma 5, consider any two points (a_1, a_2, t), $(b_1, b_2, s) \in M(W)$. Without loss of generality we can assume, that $0 < t < 1$. According to Definition 31 there exist proper binary weightings $\omega^{t'}_{a_1'a_2'}$ and $\omega^{s'}_{b_1'b_2'} \in W$ such that $\omega^{t'}_{a_1'a_2'} \rightarrow \omega^t_{a_1a_2}$ and $\omega^{s'}_{b_1'b_2'} \rightarrow \omega^s_{b_1b_2}$.

Note that for every $0 \leq a_1, a_2$, $0 \leq t \leq 1$ the weighting $\omega^t_{a_1a_2}$ by the superpositions $\omega^t_{a_1a_2}[\pi_1, \pi_1]$, $\omega^t_{a_1a_2}[\pi_2, \pi_2]$ and $\omega^t_{a_1a_2}[\wedge, \wedge]$ generates the weightings $\omega^t_{1,0}$, $\omega^t_{0,1}$ and $\omega^t_{0,0}$ respectively, and due to Lemma 10 generates both $\omega^0_{a_1a_2}$ and $\omega^1_{a_1a_2}$. The last two weightings generate the weighting $\omega^0_{1,0}$, $\omega^0_{0,1}$, $\omega^0_{0,0}$

and $\omega^1_{1,0}$, $\omega^1_{0,1}$, $\omega^1_{0,0}$ respectively. By the transitivity all these weighting are generated by the weighting $\omega^{t'}_{a'_1a'_2}$. Since for every point (a''_1, a''_2, t''), where $0 \leq a''_1, a''_2$, $a''_1 + a''_2 \leq 1$ and $0 \leq t \leq 1$ there exist nonnegative rationals $s_i, i = 1, ..., 6$, where $\sum_{i=1}^6 s_i = 1$ such that

$$\omega^{t''}_{a''_1a''_2} = s_1\omega^0_{1,0} + s_2\omega^0_{0,0} + s_3\omega^0_{0,1} + s_4\omega^1_{1,0} + s_5\omega^1_{0,0} + s_6\omega^1_{0,1}$$

then $\omega^{t'}_{a'_1,a'_2} \to \omega^{t''}_{a''_1a''_2}$ and $M(W)$ contains the whole prism with the vertices $(1, 0, 0), (0, 0, 0), (0, 1, 0), (1, 0, 1), (0, 0, 1), (0, 1, 1)$.

Finally, if $a_1 + a_2 \geq 1$, $b_1 + b_2 \geq 1$, then $\omega^t_{a_1a_2}, \omega^s_{b_1b_2} \in \mathrm{Norm}(W)$. For every $r \in [0, 1]$ the point

$$r(a_1, a_2, t) + (1-r)(b_1, b_2, s) = (ra_1 + (1-r)b_1, ra_2 + (1-r)b_2, rt + (1-r)s)$$

corresponds to the normed weighting

$$\begin{aligned}
\omega^{rt+(1-r)s}_{ra_1+(1-r)b_1,ra_2+(1-r)b_2} &= r\omega^t_{a_1a_2} + (1-r)\omega^s_{b_1b_2} = \\
&= -(ra_1 + (1-r)b_1)\pi_1 - (ra_2 + (1-r)b_2)\pi_2 + \\
&+ (r(a_1 + a_2) + (1-r)(b_1 + b_2) - 1) \wedge + \\
&+ (1 - r(t+s) - s)C_0 + (rt + (1-r)s)C_1.
\end{aligned}$$

Since W is closed under nonnegative scaling and sum of weightings, then the weighting $\omega^{rt+(1-r)s}_{ra_1+(1-r)b_1,ra_2+(1-r)b_2} \in \mathrm{Norm}(W)$ and therefore the point $(ra_1+(1-r)b_1, ra_2+(1-r)b_2, rt+(1-r)s) \in M(W)$. Together with the fact that $M(W)$ contains the prism with the vertices $(1, 0, 0), (0, 0, 0), (0, 1, 0)$, $(1, 0, 1), (0, 0, 1), (0, 1, 1)$ it follows that $M(W)$ is convex.

Now it is sufficient to note that due to Corollaries 14, 17, 18 and Lemma 10 for every (possibly improper) normed weighting $\omega^t_{a_1a_2}$ such that $(a_1, a_2, t) \in M(W)$, $M(W)$ contains all the points (a'_1, a'_2, t'), where

$(a'_1, a'_2, t') \in \{(a_1, a_2, t), (a_2, a_1, t), (a_1, 0, t), (0, a_1, t), (a_2, 0, t),$
$(0, a_2, t), (1, 0, t), (0, 1, t), (0, 0, t)\} \cup \{(a_1, a_2, 0), (a_2, a_1, 0), (a_1, 0, 0), (0, a_1, 0),$
$(a_2, 0, 0), (0, a_2, 0), (1, 0, 0), (0, 1, 0), (0, 0, 0), (a_1, a_2, 1)\} \cup \{(a_2, a_1, 1), (a_1, 0, 1)$
$(0, a_1, 1), (a_2, 0, 1), (0, a_2, 1), (1, 0, 1), (0, 1, 1), (0, 0, 1)\}.$

and therefore $M(W)$ satisfies the conditions (2), (3), (4) in the definition of $(**)$.

□

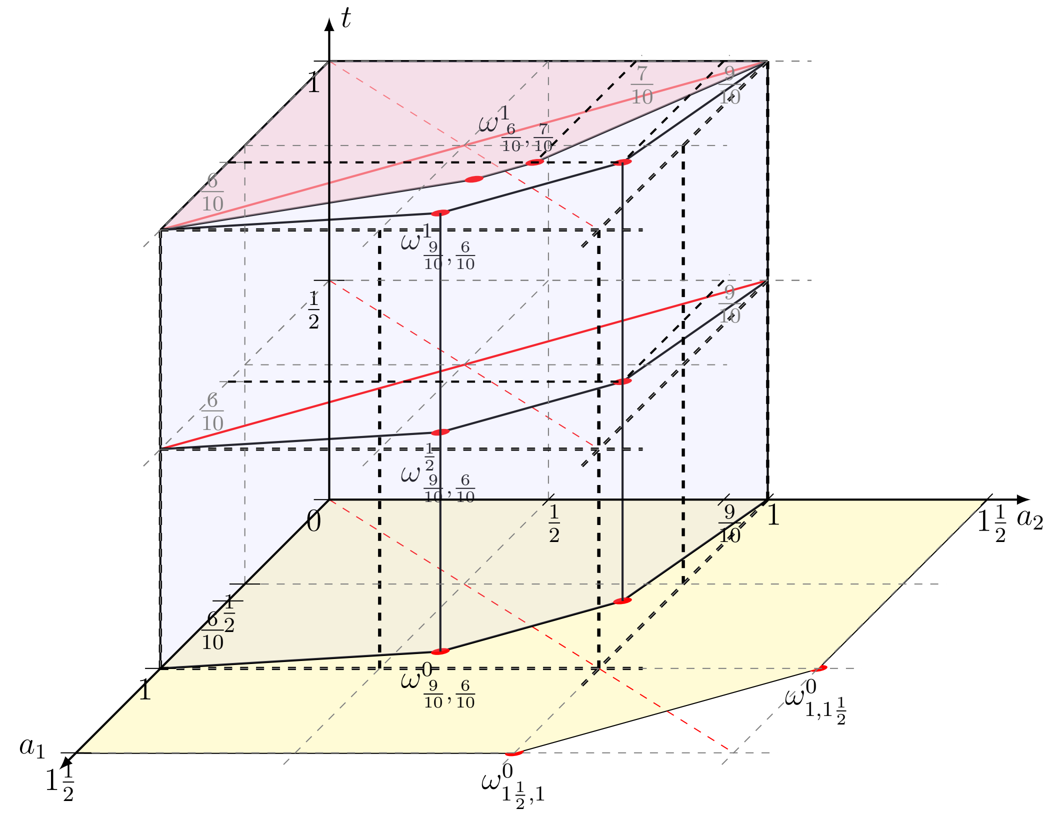

Figure 3.9: Illustration to Lemma 13.

Lemma 14. *For every set* $M \subseteq \mathbb{Q}^{t=0}_{\geq 0}$ *that satisfies Property* $(**)$ $W(M)$ *is a binary weighted clone.*

For every set $M \subseteq \mathbb{Q}^{t=1}_{\geq 0}$ *that satisfies Property* $(**)$ *and is contained in the square with vertices* $(1,0,1)$, $(0,0,1)$, $(0,1,1)$, $(1,1,1)$ *the set of weightings* $W(M)$ *is a binary weighted clone.*

For every set $M \nsubseteq \mathbb{Q}^{t=0}_{\geq 0} \cup \mathbb{Q}^{t=1}_{\geq 0}$ *that satisfies Property* $(**)$ *and is contained in the cube with vertices* $(1,0,0)$, $(0,0,0)$, $(0,1,0)$, $(1,1,0)$, $(1,0,1)$, $(0,0,1)$, $(0,1,1)$, $(1,1,1)$ *the set of weightings* $W(M)$ *is a binary weighted clone.*

Proof. First, note that due to Corollaries 17 and 18 and Lemmas 6 and 9 we do not need to prove first two items of the statement, because for planes $\mathbb{Q}^{t=0}_{\geq 0}$ and $\mathbb{Q}^{t=1}_{\geq 0}$ satisfying Property $(**)$ is the same as satisfying Property $(*)$.

Thus, we only have to prove the third item of the statement. Consider a proper binary weighting ω which is equal to

$$\omega = r_1\omega_1[f_1, g_1] + r_2\omega_2[f_2, g_2] + ... + r_n\omega_n[f_n, r_n]$$

where $r_1, r_2, ..., r_n$ are nonnegative rationals, $\omega_1, \omega_2, ..., \omega_n \in W(M)$ and $f_i, g_i, i = 1, ...n$ are binary operations from the clone $\wedge P_{01}$. According to Definition 32, each $\omega_i, i = 1, ..., n$ is of the form $s_i\omega^{t_i}_{a_i b_i} : (a_i, b_i, t_i) \in M, s \geq 0, a_i + b_i \geq 1$, where $(a_i, b_i, t_i) \in M$ and $0 \leq t \leq 1$. By Corollaries 14, 17 and 18, for each binary operations f_i, g_i from $\wedge P_{01}$, $\omega^{t_i}_{a_i b_i}[f_i, g_i]$ is either the zero weighting, or a positive multiple of the weightings $\theta = -C_0 + C_1$, $\eta = C_0 - C_1$ or $\omega^{t'_i}_{a'_i b'_i}$, where $(a'_i, b'_i, t'_i) \in P$. Assume without loss of generality that $\omega^{t_i}_{a_i b_i}[f_i, g_i]$ for every $i = 1, ..., n$ is not the zero weighting. Therefore, after renumbering of weightings $\omega_i, ..., \omega_n$ for some $m \leq n$ and $x, y \geq 0$, we can rewrite ω as

$$\omega = r_1 s_1 k_1 \omega^{t'_1}_{a'_1,b'_1} + ... + r_m s_m k_m \omega^{t'_m}_{a'_m b'_m} + x(C_0 - C_1) + y(-C_0 + C_1) =$$
$$(r_1 s_1 k_1 + ... + r_m s_m k_m)(p_1 \omega^{t'_1}_{a'_1,b'_1} + ... + p_m \omega^{t'_m}_{a'_m b'_m}) +$$
$$+ x(C_0 - C_1) + y(-C_0 + C_1) = s\omega^t_{ab} + x(C_0 - C_1) + y(-C_0 + C_1),$$

where $s = (r_1 s_1 k_1 + ... + r_m s_m k_m)$, $(p_1 + ... + p_m) = 1$, $a = (p_1 a'_1 + ... + p_m a'_m), b = (p_1 b'_1 + ... + p_m b'_m)$, $t = (p_1 t'_1 + ... + p_m t'_m)$. Since M is convex, then $(a, b, t) \in M$ and $s\omega^t_{ab} \in W(M)$. Since ω is proper weighting, then the weights on the constant are nonnegative, i.e.

$$0 \leq st - x + y \Rightarrow 0 \leq t - (\frac{x}{s} - \frac{y}{s}),$$
$$0 \leq s(1 - t) - x + y \Rightarrow 0 \leq 1 - t - \frac{x}{s} + \frac{y}{s} \Rightarrow t - (\frac{x}{s} - \frac{y}{s}) \leq 1.$$

Thus, the point (a, b, t'), where $t' = t - (\frac{x}{s} - \frac{y}{s})$, is still in M, and therefore the weighting $\omega = s\omega^{t'}_{ab}$ is in $M(W)$. According to Definition 23, $W(M)$ is binary weighted clone over the clone $\wedge P_0$. □

As in two previous sections we first formulate some results, that will help us to construct the whole lattice of binary weighted clones over the clone $\wedge P_{01}$.

Theorem 12. *There are two nontrivial atomic weighted clones over the clone $\wedge P_{01}$:*

(1) The weighted clone $W^{t=0}_{1,0}$, defined as follows: for every k

$$(W^{t=0}_{1,0})^k = \{\omega \in W_{\wedge P_{01}} : \textit{ for every } I \subseteq \{1, ..., k\} \textit{ s. t. } |I| > 1,$$
$$\omega(\wedge_I) = \omega(C_1) = 0\}.$$

Moreover, $W^{t=0}_{1,0}$ is generated by the unary weighting $\omega^0_{1,0} = -\pi_1 + C_0$.

(2) The weighted clone $W^{t=1}_{1,0}$, defined as follows: for every k

$$(W^{t=1}_{1,0})^k = \{\omega \in W_{\wedge P_{01}} : \textit{ for every } I \subseteq \{1, ..., k\} \textit{ s. t. } |I| > 1,$$
$$\omega(\wedge_I) = \omega(C_0) = 0\}.$$

Moreover, $W^{t=1}_{1,0}$ is generated by the unary weighting $\omega^1_{1,0} = -\pi_1 + C_1$.

There are the smallest weighted clone $W^{0<t<1}_{1,0}$ containing both $W^{t=0}_{1,0}$ and $W^{t=1}_{1,0}$, defined as follows: for every k

$$(W^{0<t<1}_{1,0})^k = \{\omega \in W_{\wedge P_{01}} : \textit{ for every } I \subseteq \{1, ..., k\} \textit{ s. t. } |I| > 1, \ \omega(\wedge_I) = 0\}.$$

Moreover, $W^{0<t<1}_{1,0}$ is generated by the unary weighting $\omega^{\frac{1}{2}}_{1,0} = -\pi_1 + \frac{1}{2}C_0 + \frac{1}{2}C_1$.

Proof. For the first part of the theorem, note that the proof of the fact, that $W^0_{1,0} = wClone(\omega^0_{1,0})$ and $W^1_{1,0} = wClone(\omega^1_{1,0})$, i.e. that $W^0_{1,0}$ and $W^1_{1,0}$ are actually weighted clones and are generated by the weightings $\omega^0_{1,0}$ and $\omega^1_{1,0}$ respectively, is the same as the corresponding part of the proofs of Theorems 6, 9. Thus, we only have to show that these weighted clones are atomic (there are no smaller weighted clones, contained in $W^0_{1,0}$ or $W^1_{1,0}$), and there is no others atomic weighted clones, that is, every nontrivial proper weighting generates either both $\omega^0_{1,0}$ and $\omega^1_{1,0}$ or one of them. Indeed, for every k-ary and m-ary weightings $\tau \in W^0_{1,0}$ and $\theta \in W^1_{1,0}$ superpositions

$$\frac{1}{\sum_{i=1}^{k} \tau(\pi_i^k)} \tau[\pi_1^k, ..., \pi_1^k] = \omega^0_{1,0} \qquad and \qquad \frac{1}{\sum_{i=1}^{m} \theta(\pi_i^m)} \theta[\pi_1^m, ..., \pi_1^m] = \omega^1_{1,0}.$$

Therefore, the weighted clones $W^0_{1,0}$, $W^1_{1,0}$ are atomic. Next, consider an arbitrary weighting $\omega^t_{a_1,a_2}$. If $t = 0$ or $t = 1$, then, due to Lemmas 4 or 7, $\omega^t_{a_1,a_2}$ generates the weighting $\omega^0_{1,0}$ or the weighting $\omega^1_{1,0}$ respectively. If $0 < t < 1$, then, due to Lemma 10, $\omega^t_{a_1,a_2}$ generates both weightings $\omega^0_{1,0}$ and $\omega^1_{1,0}$. Thus, there is no other atomic weighted clones over the clone $\wedge P_{01}$.

To prove the second part of the theorem we first show that $W^{\frac{1}{2}}_{1,0}$ is a weighted clone generated by $\omega^{\frac{1}{2}}_{1,0}$, i.e. $W^{\frac{1}{2}}_{1,0} = wClone(\omega^{\frac{1}{2}}_{1,0})$. On the one hand, each k-ary weighting ω from $W^{\frac{1}{2}}_{1,0}$ is of the form

$$\begin{aligned}
&\omega = -a_1\pi_1 - a_2\pi_2 - ... - a_k\pi_k + c_0C_0 + c_1C_1 = \\
&- a_1\pi_1 - a_2\pi_2 - ... - a_k\pi_k + c_0C_0 + (a_1 + a_2 + ... + a_k - c_0)C_1 = \\
&a_1(-\pi_1 + \frac{1}{2}C_0 + \frac{1}{2}C_1) + ... + a_k(-\pi_k + \frac{1}{2}C_0 + \frac{1}{2}C_1) + \\
&+ \frac{1}{2}(\sum_{i=1}^{k} a_i)C_0 - \frac{1}{2}(\sum_{i=1}^{k} a_i)C_1 - (\sum_{i=1}^{k} a_i - c_0)C_0 + (\sum_{i=1}^{k} a_i - c_0)C_1 = \\
&a_1\omega^{\frac{1}{2}}_{1,0}[\pi_1] + a_2\omega^{\frac{1}{2}}_{1,0}[\pi_2] + ... + a_k\omega^{\frac{1}{2}}_{1,0}[\pi_k] + (\sum_{i=1}^{k} a_i)\omega^{\frac{1}{2}}_{1,0}[C_1] + 2(\sum_{i=1}^{k} a_i - c_0)\omega^{\frac{1}{2}}_{1,0}[C_0].
\end{aligned}$$

Therefore, $W^{\frac{1}{2}}_{1,0} \subseteq wClone(\omega^{\frac{1}{2}}_{1,0})$. On the other hand, due to Lemma 2 for every proper k-ary weighting $\omega \in wClone(\omega^{\frac{1}{2}}_{1,0})$ there exist nonnegative rationals $a_1, ..., a_k$, b_I for $I \subseteq \{1, ..., k\}, |I| > 1$ and c_0, c_1 such that ω is equal to:

$$\begin{aligned}
&\omega = a_1\omega^{\frac{1}{2}}_{1,0}[\pi_1] + ... + a_k\omega^{\frac{1}{2}}_{1,0}[\pi_k] + \sum_{\substack{I\subseteq\{1,...,k\}\\|I|>1}} b_I\omega^{\frac{1}{2}}_{1,0}[\wedge_I] + c_0\omega^{\frac{1}{2}}_{1,0}[C_0] + c_1\omega^{\frac{1}{2}}_{1,0}[C_1] = \\
&a_1(-\pi_1 + \frac{1}{2}C_0 + \frac{1}{2}C_1) + ... + a_k(-\pi_k + \frac{1}{2}C_0 + \frac{1}{2}C_1) + \\
&+ \sum_{\substack{I\subseteq\{1,...,k\}\\|I|>1}} b_I(- \wedge_I + \frac{1}{2}C_0 + \frac{1}{2}C_1) + c_0(-\frac{1}{2}C_0 + \frac{1}{2}C_1) + c_1(\frac{1}{2}C_0 - \frac{1}{2}C_1) = \\
&- a_1\pi_1 - ... - a_k\pi_k - \sum_{\substack{I\subseteq\{1,...,k\}\\|I|>1}} b_I \wedge_I + \\
&+ \frac{1}{2}(\sum_{i=1}^{k} a_i + \sum_{\substack{I\subseteq\{1,...,k\}\\|I|>1}} b_I - c_0 + c_1)C_0 + \frac{1}{2}(\sum_{i=1}^{k} a_i + \sum_{\substack{I\subseteq\{1,...,k\}\\|I|>1}} b_I + c_0 - c_1)C_1.
\end{aligned}$$

Since ω is proper, we have $b_I = 0$ for all $I \subseteq \{1,...,k\}, |I| > 1$. Thus, every k-ary weighting $\omega \in wClone(\omega_{1,0}^{\frac{1}{2}})$ is in $W_{1,0}^{\frac{1}{2}}$ and $wClone(\omega_{1,0}^{\frac{1}{2}}) \subseteq W_{1,0}^{\frac{1}{2}}$. Therefore $wClone(\omega_{1,0}^{\frac{1}{2}}) = W_{1,0}^{\frac{1}{2}}$.

Now suppose that there is a smaller weighted clone W that contains both $W_{1,0}^0$ and $W_{1,0}^1$. Then the weightings $\omega_{1,0}^0$ and $\omega_{1,0}^0$ are in W. But since $\frac{1}{2}(\omega_{1,0}^0 + \omega_{1,0}^0) = \omega_{1,0}^{\frac{1}{2}}$, then $W_{1,0}^{\frac{1}{2}} \subseteq W$ and it is contradiction. □

Theorem 13. *The nontrivial weighted clone over the clone* $\wedge P_{01}$ *that contains all others nontrivial weighted clones consisting of weightings with zero weight on the constant* C_1*, except the weighted clone* $W_\wedge$*, is* $W_{C_0 \neq 0}$*, defined as follows: a* k*-ary weighting* ω *is in* $W_{C_0\neq 0}$ *if and only if for each nonempty set of coordinates* $I \subseteq \{1,...,k\}$ *and for each nonempty set of coordinates* $T \subseteq \{1,...,k\}$ *such that* $T \cap \{j : \omega(\pi_j^k) < 0\} \neq \emptyset$,

$$\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J^k) \leq 0, \tag{3.16}$$

$$\sum_{\emptyset \neq J \subseteq T} \omega(\wedge_J^k) < 0, \tag{3.17}$$

$$\omega(C_1) = 0. \tag{3.18}$$

Proof. The proof exactly coincides with the proof of Theorem 7, except for the one point: we have to prove that for every weighting $\tau \in W_{C_0\neq 0}$ with any superposition we cannot generate the positive weight on the constant C_1. But it follows immediately from transitivity of the relation $\rightarrow$ and the fact that we cannot get positive weight on C_1 by any superposition of the weighting $\omega_\wedge$. □

Theorem 14. *The nontrivial weighted clone over the clone* $\wedge P_{01}$ *that contains all others nontrivial weighted clones consisting of weightings with zero weight on the constant* C_0*, except the weighted clone* $W_\wedge$*, is* $W_{\leq -C_1}$*, defined as follows: a* k*-ary weighting* ω *is in* $W_{\leq -C_1}$ *if and only if for each* $j \in \{1,...,k\}$ *and for each nonempty set of coordinates* $\emptyset \neq K \subseteq I \subseteq \{1,...,k\}$

$$\omega(\pi_j) \leq 0, \tag{3.19}$$

$$\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J) \leq \sum_{\emptyset \neq J \subseteq K} \omega(\wedge_J), \tag{3.20}$$

$$\omega(C_0) = 0. \tag{3.21}$$

Proof. Again, the proof exactly coincides with the proof of Theorem 10, except for the one point: we have to prove that for every weighting $\tau \in W_{\leq -C_1}$ with any superposition we cannot generate the positive weight on the constant C_0. Suppose that there exist a k-ary superposition $\omega = \tau[f_1, ..., f_m]$ of $\tau = \sum_{\emptyset \neq I \subseteq \{1,...,m\}} \tau(\wedge_I^m) \wedge_I^m + cC_1 \in W_{\leq -C_1}^m$ such that $\omega(C_0) > 0$. We know that

$$\omega(C_0) = \tau[f_1, ..., f_m](C_0) = \sum_{\substack{I' \subseteq \{1,...,m\} \\ I' \cap \{j : f_j = C_0\} \neq \emptyset}} \tau(\wedge_{I'}^m) > 0,$$

denote $\{j : f_j = C_0\} = K$. But

$$\sum_{\substack{I' \subseteq \{1,...,m\} \\ I' \cap K \neq \emptyset}} \tau(\wedge_{I'}^m) = \sum_{\emptyset \neq J \subseteq \{1,...,m\}} \omega(\wedge_J) - \sum_{\emptyset \neq J \subseteq \{1,...,m\} \setminus K} \omega(\wedge_J) \leq 0,$$

since τ satisfies condition (3.20). This is a contradiction and we are done. □

Theorem 15. *There are two maximal weighted clones over the clone* $\wedge P_{01}$*:*

(1) *the weighted clone* $W_\wedge$ *that contains all others weighted clones of weightings with zero weight on the constant* C_1*, defined as follows: a k-ary weighting* ω *is in* $W_\wedge$ *if and only if* $\omega(C_1) = 0$ *and for each nonempty set of coordinates* $I \subseteq \{1, ..., k\}$

$$\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J) \leq 0. \tag{3.22}$$

Moreover, $W_\wedge$ *is generated by the weighting* $\omega_\wedge = -\pi_1 - \pi_2 + 2\wedge$*;*

(2) *the weighted clone* $W_{C_0\neq 0 \cup C_1 \neq 0}$ *defined as follows: a k-ary weighting* ω *is in* $W_{C_0\neq 0 \cup C_1 \neq 0}$ *if and only if for each* $j \in \{1, ..., k\}$ *and each nonempty set of coordinates* $\emptyset \neq K \subseteq I \subseteq \{1, ..., k\}$

$$\omega(\pi_j) \leq 0, \tag{3.23}$$

$$\sum_{\emptyset \neq J \subseteq I} \omega(\wedge_J) \leq \sum_{\emptyset \neq J \subseteq K} \omega(\wedge_J). \tag{3.24}$$

Proof. We only prove the second part of the theorem, because the first part follows immediately from Theorem 6 and Corollary 16.

At first, we prove an auxiliary statement.

Claim 2. *A k-ary weighting* ω *is in* $W_{C_0\neq 0 \cup C_1 \neq 0}$ *if and only if for each binary operations* $g_1, g_2, ..., g_k$ *from* $(\wedge P_{01})^2$ *a weighting* $\phi = \omega[g_1, g_2, ..., g_k]$ *for* $i = 1, 2$ *satisfies condition*

$$-\phi(C_0) - \phi(C_1) \leq \phi(\pi_i) \leq 0. \tag{3.25}$$

Proof of Claim 2. Consider an arbitrary k-ary weightings $\omega \in W_{C_0\neq 0 \cup C_1 \neq 0}$ and the superposition $\phi = \omega[g_1, g_2, ..., g_k]$, where $g_1, g_2, ..., g_k \in (\wedge P_{01})^2$. There are five binary operations containing in $(\wedge P_{01})$: the two projections π_1, π_2, the meet operation $\wedge$ and the constant operations C_0, C_1. Let $T_{C_0} = \{j : g_j = C_0\}$, $T_{C_1} = \{j : g_j = C_1\}$, $T_{\pi_i} = \{j : g_j = \pi_i\}$. Note that

$$\phi(\pi_i) = \sum_{\substack{\emptyset \neq J \subseteq T_{C_1} \cup T_{\pi_i}, \\ J \cap T_{\pi_i} \neq \emptyset}} \omega(\wedge_J) = \sum_{\emptyset \neq J \subseteq T_{C_1} \cup T_{\pi_i}} \omega(\wedge_J) - \sum_{\emptyset \neq J \subseteq T_{C_1}} \omega(\wedge_J) \leq 0,$$

since ω satisfies condition (3.24) and

$$\phi(C_0) = \sum_{\substack{\emptyset \neq J \subseteq \{1,...,k\}, \\ J \cap T_{C_0} \neq \emptyset}} \omega(\wedge_J) + \omega(C_0) =$$

$$= \sum_{\emptyset \neq J \subseteq \{1,...,k\}} \omega(\wedge_J) - \sum_{\emptyset \neq J \subseteq \{1,...,k\} \setminus T_{C_0}} \omega(\wedge_J) + \omega(C_0),$$

$$\phi(C_1) = \sum_{\emptyset \neq J \subseteq T_{C_1}} \omega(\wedge_J) + \omega(C_1)$$

Therefore,

$$\phi(\pi_i) + \phi(C_0) + \phi(C_1) = \sum_{\emptyset \neq J \subseteq T_{C_1} \cup T_{\pi_i}} \omega(\wedge_J) - \sum_{\emptyset \neq J \subseteq T_{C_1}} \omega(\wedge_J) +$$

$$+ \sum_{\emptyset \neq J \subseteq \{1,...,k\}} \omega(\wedge_J) - \sum_{\emptyset \neq J \subseteq \{1,...,k\} \setminus T_{C_0}} \omega(\wedge_J) + \omega(C_0) + \omega(C_1) + \sum_{\emptyset \neq J \subseteq T_{C_1}} \omega(\wedge_J).$$

Note that $\omega(C_0)+\omega(C_1) = -\sum_{\emptyset\neq J\subseteq\{1,...,k\}}\omega(\wedge_J)$ and $T_{C_1}\cup T_{\pi_i}\subseteq\{1,...,k\}\setminus T_{C_0}$. Then

$$\phi(\pi_i)+\phi(C_0)+\phi(C_1) = \sum_{\emptyset\neq J\subseteq T_{C_1}\cup T_{\pi_i}}\omega(\wedge_J) - \sum_{\emptyset\neq J\subseteq\{1,...,k\}\setminus T_{C_0}}\omega(\wedge_J)\geq 0,$$

again since ω satisfies condition (3.24). Thus, ϕ satisfies conditions (3.25).

On the other hand, assume that proper weighting ω is not in $\omega\in W_{C_0\neq 0\cup C_1\neq 0}$. If ω violates condition (3.23), then there exists $j\in\{1,...,k\}$ such that $\omega(\pi_j)>0$. Then the superposition

$$\phi = \omega[\pi_1,\pi_1,...\pi_2,...,\pi_1]$$

with the second projection on the j-th coordinate violates condition (3.25), since $\phi(\pi_2)>0$. If ω violates condition (3.24), then there exist two nonempty sets $\emptyset\neq K\subseteq I\subseteq\{1,...,k\}$, such that $\sum_{\emptyset\neq J\subseteq I}\omega(\wedge_J) > \sum_{\emptyset\neq J\subseteq K}\omega(\wedge_J)$. Without loss of generality we can assume that $K=\{1,...,p\}$, $I=\{1,...,p+t\}$ for some $0<p,t$. Then superposition

$$\phi = \omega[\underbrace{C_1,...,C_1}_{p},\underbrace{\pi_1,...,\pi_1}_{t},g_{(p+t+1)},...,g_m]$$

violates condition (3.25), since

$$\phi(\pi_1) = \sum_{\emptyset\neq J\subseteq\{1,...,(p+t)\}}\omega(\wedge_J) - \sum_{\emptyset\neq J\subseteq\{1,...,p\}}\omega(\wedge_J) > 0.$$

Let us prove that $W_{C_0\neq 0\cup C_1\neq 0}$ is a weighted clone. $W_{C_0\neq 0\cup C_1\neq 0}$ is closed under conical combinations of weightings, thus, we only have to prove that $W_{C_0\neq 0\cup C_1\neq 0}$ is closed under proper superposition. But it is an easy consequence from Lemma 2. Therefore, $W_{C_0\neq 0\cup C_1\neq 0}$ is a weighted clone.

It remains to prove that $W_{C_0\neq 0\cup C_1\neq 0}$ is maximal. Suppose that there exist nontrivial weighted clone W containing $W_{C_0\neq 0\cup C_1\neq 0}$. Since $W\neq W_{C_0\neq 0\cup C_1\neq 0}$, then W contains nonzero k-ary weighting ω that violates conditions (3.23), (3.24). If ω violates condition (3.23), then $W = W_{\wedge P_{01}}$ and it is contradiction. Suppose that ω violates condition (3.24), there exist two nonempty sets $\emptyset\neq K\subseteq I\subseteq\{1,...,k\}$, such that $\sum_{\emptyset\neq J\subseteq I}\omega(\wedge_J) > \sum_{\emptyset\neq J\subseteq K}\omega(\wedge_J)$, $K=\{1,...,p\}$, $I=\{1,...,p+t\}$. The conical combination

$$\theta = \omega[\underbrace{C_1,...,C_1}_{p},\underbrace{\pi_2,...,\pi_2}_{t},\pi_{(p+t+1)},...,\pi_m]$$

has a positive weight on the projection π_2, but might have a negative weight on the constant C_1. We can fix it by the following way. Note that the weighting $\omega^1_{1,0} = -\pi_1 + C_1$ is in $W_{C_0\neq 0\cup C_1\neq 0}$. Since $W_{C_0\neq 0\cup C_1\neq 0}\subsetneq W$, then $\omega^1_{1,0}\in W$. Now consider a conical combination

$$\theta' = \omega[\underbrace{C_1,...,C_1}_{p},\underbrace{\pi_2,...,\pi_2}_{t},\pi_{(p+t+1)},...,\pi_m] + \Big(\sum_{\emptyset\neq J\subseteq\{1,...,p\}}\omega(\wedge_J)+\omega(C_1)\Big)\omega^1_{0,1},$$

which is proper and has a positive weight on the projection π_2. Thus $W = W_{\wedge P_{01}}$, and it is contradiction. Therefore, $W_{C_0\neq 0\cup C_1\neq 0}$ is maximal.

□

We are finally ready to formulate the theorem that describe all binary weighted clones over the clone $\wedge P_{01}$.

Theorem 16. *Every nontrivial binary weighted clone over the clone $\wedge P_{01}$ is either $\mathbf{BP}(W_\wedge)$ or is equal to $W(M)$ for some M such that:*

(1) $M \subseteq \mathbb{Q}_{\geq 0}^{t=0}$ and satisfies Property $(**)$ or

(2) $M \subseteq \mathbb{Q}_{\geq 0}^{t=1}$ is contained in the square with vertices $(1,0,1), (0,0,1)$, $(0,1,1)$, $(1,1,1)$ and satisfies Property $(**)$ or

(3) $M \subseteq \mathbb{Q}_{\geq 0}^{3}$ is contained in the cube with vertices $(1,0,0), (0,0,0)$, $(0,1,0)$, $(1,1,0)$, $(1,0,1)$, $(0,0,1), (0,1,1), (1,1,1)$ and satisfies Property $(**)$.

For every such set M $W(M) \subsetneq \mathbf{BP}(W_\wedge)$ if and only if $M \subseteq \mathbb{Q}_{\geq 0}^{t=0}$. For every two such sets M, M' $W(M) \subseteq W(M')$ if and only if $M \subseteq M'$.

There is continuum many binary weighted clones. These binary weighted clones are generated by different (infinite) sets of normed binary weightings.

Proof. Consider an arbitrary binary weighted clone W. We know that every nonzero binary weighting with the zero weights on the both constants C_0, C_1 generates $\omega_\wedge$. Thus, if W contains ω, then W is the binary part of $W_\wedge$. Therefore, it is sufficient to consider an arbitrary binary weighted clone W over the clone $\wedge P_{01}$, that contains only binary weightings with nonzero weight on the constants (except the zero weighting).

Due to Lemma 13, the set $M(W)$ satisfies $(**)$. If W contains only binary weighting with nonzero weight on the constant C_0 and zero weight on the constant C_1, then $M(W) \subseteq \mathbb{Q}_{\geq 0}^{t=0}$. From Lemma 11 we know that if W contains the weighting ω such that, being normed, has coefficient $0 < t \leq 1$ and at least one of two coefficients a_1, a_2 greater than 1, then W is a binary part of $W_{\wedge P_{01}}$. Therefore, it is sufficient to consider a binary weighted clone W, that contains only binary weighting ω with nonzero weight on the constant C_1 and zero weight on the constant C_0 such that, being normed, have coefficients $a_1, a_2 \leq 1$, and a binary weighted clone W, that contains binary weighting ω such that $\omega(C_0)+\omega(C_1) > 0$ and ω, being normed, has coefficients $a_1, a_2 \leq 1$. In first case $M(W) \subseteq \mathbb{Q}_{\geq 0}^{t=1}$ and is contained in the square with vertices $(1,0,1), (0,0,1)$, $(0,1,1)$,$(1,1,1)$, and in the second case $M(W) \subseteq \mathbb{Q}_{\geq 0}^{3}$ is contained in the cube with vertices $(1,0,0), (0,0,0)$, $(0,1,0)$, $(1,1,0)$,$(1,0,1)$,$(0,0,1)$, $(0,1,1)$, $(1,1,1)$.

We have to prove now that in the all these cases $W(M(W)) = W$. Consider an arbitrary nonzero binary weighting $\omega \in W$. Since $\omega(C_0) + \omega(C_1) \neq 0$ then $\omega = s\omega_{a_1a_2}^t$ for some nonnegative rationals s, a_1, a_2, $0 < t < 1$ and therefore $(a_1, a_2, t) \in M(W)$. But since ω is proper then $a_1 + a_2 \geq 1$ and thus $\omega = s\omega_{a_1a_2}^t \in W(M(W))$. On the other hand, according to Definition 32 each binary weighting $\omega \in W(M(W))$ is proper and of the form $s\omega_{a_1a_2}^t$ for some $s \geq 0$ and $(a_1, a_2, t) \in M(W)$, where $a_1 + a_2 \geq 1$, $0 < t < 1$, which implies that $\omega_{a_1a_2}^t \in \mathrm{Norm}(W)$. Since W is closed under nonnegative scaling, then $\omega = s\omega_{a_1a_2}^t \in W$. Therefore, $W(M(W)) = W$.

To prove the rest note that $\mathbf{BP}(W_\wedge)$ contains all other binary weighted clones with zero weight on the constant C_1 and $M(\mathbf{BP}(W_\wedge))$ is the whole $\mathbb{Q}_{\geq 0}^{t=0}$. From Corollary 16 we know, that if W contains both a nonzero weighting with zero weight on the constant C_1 and a nonzero weighting with positive weight on the constant C_1, then W is the binary part of the trivial weighted clone $\wedge P_1$. Hence, if $M \subseteq \mathbb{Q}^{t=1}_{\geq 0}$ then $W(M)$ is incomparable with $\mathbf{BP}(W_\wedge)$. Finally, from Lemma 13 we know, that if $M \nsubseteq \mathbb{Q}^{t=0}_{\geq 0}$ and $M \nsubseteq \mathbb{Q}^{t=1}_{\geq 0}$ then M intersect with $\mathbb{Q}^{t=0}_{\geq 0}$ and due to Lemma 14 $W(M \cap \mathbb{Q}^{t=0}_{\geq 0})$ is a weighted clone. Finally, note that if $M \subseteq M'$, then $W(M) \subseteq W(M')$ directly from the definition of $W(M)$. Otherwise, if there exist a point $(a_1, a_2, t) \in M$ such that $(a_1, a_2, t) \notin M'$, then, for every $s \geq 0$, the weighting $s\omega^t_{a_1a_2}$ is in $W(M)$ but not in $W(M')$. Therefore $W(M) \nsubseteq W(M')$. $\square$

To illustrate the lattice of binary weighted clones over the clone $\wedge P_{01}$ we introduce the following notation. Given a binary weighting in normed form $\omega^t_{a_1a_2}$ we denote the binary weighted clone generated by this weighting by $W^{t=0}_{a_1a_2}$, if $t = 0$, by $W^{t=1}_{a_1a_2}$, if $t = 1$ and $W^{0<t<1}_{a_1a_2}$ otherwise (Given a set of binary weightings in normed form $\omega^t_{a_1a_2}$, $\omega^t_{b_1b_2}$, ... we denote the binary weighted clone, generated by those weightings by $W^{t=0}_{a_1a_2,b_1b_2,...}$, $W^{t=1}_{a_1a_2,b_1b_2,...}$ and $W^{0<t<1}_{a_1a_2,b_1b_2,...}$ respectively).

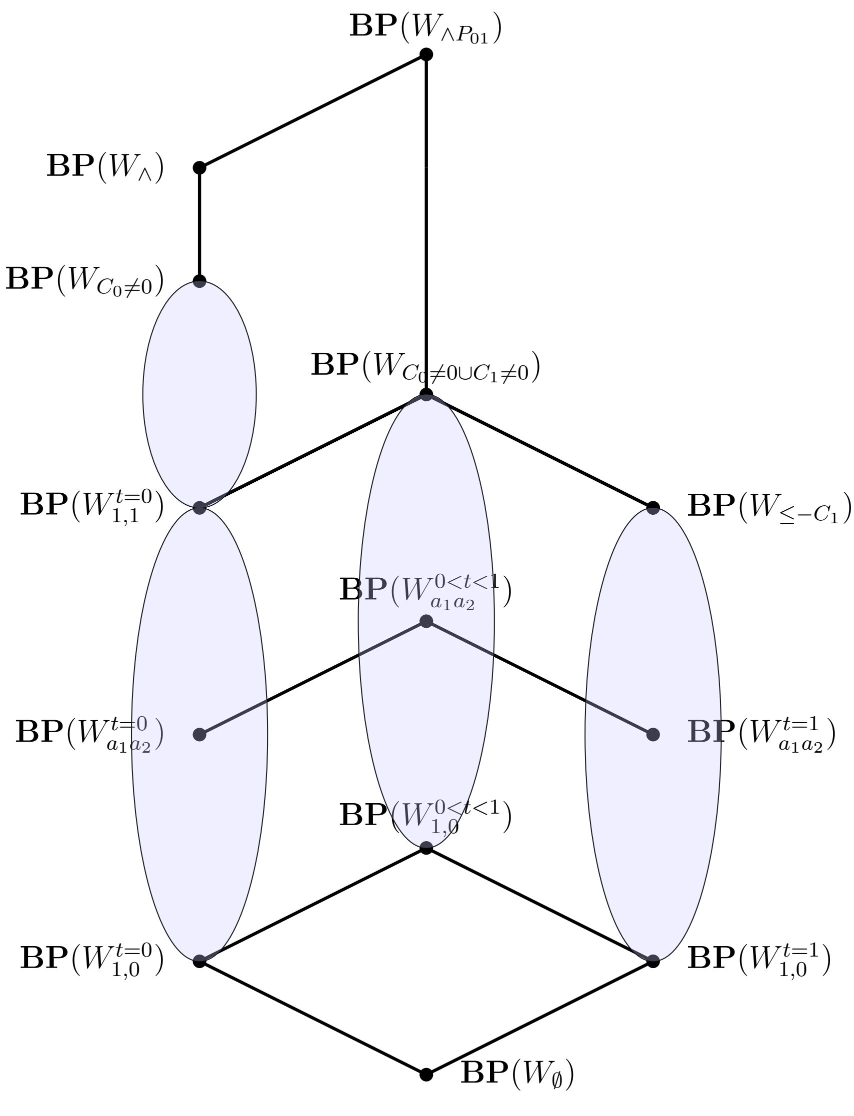

Figure 3.10: The lattice of binary weighted clones over the clone $\wedge P_{01}$.

The dual clone $\vee P_{01}$ has the similar structure: in all claims we just switch the constants C_0 to C_1 and vise versa.

Corollary 19. *There are two nontrivial atomic weighted clones over the clone $\vee P_{01}$:*

(1) The weighted clone $W_{1,0}^{t=0}$, defined as follows: for every k

$$(W_{1,0}^{t=0})^k = \{\omega \in W_{\vee P_{01}} : \textit{ for every } I \subseteq \{1, ..., k\} \textit{ s. t. } |I| > 1, \\ \omega(\vee_I) = \omega(C_1) = 0\}.$$

Moreover, $W_{1,0}^{t=0}$ is generated by the unary weighting $\omega_{1,0}^0 = -\pi_1 + C_0$.

(2) The weighted clone $W_{1,0}^{t=1}$, defined as follows: for every k

$$(W_{1,0}^{t=1})^k = \{\omega \in W_{\vee P_{01}} : \textit{ for every } I \subseteq \{1, ..., k\} \textit{ s. t. } |I| > 1, \\ \omega(\vee_I) = \omega(C_0) = 0\}.$$

Moreover, $W_{1,0}^{t=1}$ is generated by the unary weighting $\omega_{1,0}^1 = -\pi_1 + C_1$.

There are the smallest weighted clone $W_{1,0}^{0<t<1}$ containing both $W_{1,0}^{t=0}$ and $W_{1,0}^{t=1}$, defined as follows: for every k

$$(W_{1,0}^{0<t<1})^k = \{\omega \in W_{\vee P_{01}} : \textit{ for every } I \subseteq \{1, ..., k\} \textit{ s. t. } |I| > 1, \ \omega(\vee_I) = 0\}.$$

Moreover, $W_{1,0}^{0<t<1}$ is generated by the unary weighting $\omega_{1,0}^{\frac{1}{2}} = -\pi_1 + \frac{1}{2}C_0 + \frac{1}{2}C_1$.

Corollary 20. *The nontrivial weighted clone over the clone $\vee P_{01}$ that contains all others nontrivial weighted clones consisting of weightings with zero weight on the constant C_0, except the weighted clone $W_\vee$, is $W_{C_1 \neq 0}$, defined as follows: a k-ary weighting ω is in $W_{C_1 \neq 0}$ if and only if for each nonempty set of coordinates $I \subseteq \{1, ..., k\}$ and for each nonempty set of coordinates $T \subseteq \{1, ..., k\}$ such that $T \cap \{j : \omega(\pi_j^k) < 0\} \neq \emptyset$,*

$$\sum_{\emptyset \neq J \subseteq I} \omega(\vee_J^k) \leq 0, \tag{3.26}$$

$$\sum_{\emptyset \neq J \subseteq T} \omega(\vee_J^k) < 0, \tag{3.27}$$

$$\omega(C_0) = 0. \tag{3.28}$$

Corollary 21. *The nontrivial weighted clone over the clone $\vee P_{01}$ that contains all others nontrivial weighted clones consisting of weightings with zero weight on the constant C_1, except the weighted clone $W_\vee$, is $W_{\leq -C_0}$, defined as follows: a k-ary weighting ω is in $W_{\leq -C_0}$ if and only if for each $j \in \{1, ..., k\}$ and for each nonempty set of coordinates $\emptyset \neq K \subseteq I \subseteq \{1, ..., k\}$*

$$\omega(\pi_j) \leq 0, \tag{3.29}$$

$$\sum_{\emptyset \neq J \subseteq I} \omega(\vee_J) \leq \sum_{\emptyset \neq J \subseteq K} \omega(\vee_J), \tag{3.30}$$

$$\omega(C_1) = 0. \tag{3.31}$$

Corollary 22. *There are two maximal weighted clones over the clone* $\vee P_{01}$*:*

(1) the weighted clone $W_\vee$ *that contains all others weighted clones of weightings with zero weight on the constant* C_0*, defined as follows: a k-ary weighting* ω *is in* $W_\vee$ *if and only if* $\omega(C_0) = 0$ *and for each nonempty set of coordinates* $I \subseteq \{1, ..., k\}$

$$\sum_{\emptyset \neq J \subseteq I} \omega(\vee_J) \leq 0. \tag{3.32}$$

Moreover, $W_\vee$ *is generated by the weighting* $\omega_\vee = -\pi_1 - \pi_2 + 2\vee$*;*

(2) the weighted clone $W_{C_0 \neq 0 \cup C_1 \neq 0}$ *defined as follows: a k-ary weighting* ω *is in* $W_{C_0 \neq 0 \cup C_1 \neq 0}$ *if and only if for each* $j \in \{1, ..., k\}$ *and each nonempty set of coordinates* $\emptyset \neq K \subseteq I \subseteq \{1, ..., k\}$

$$\omega(\pi_j) \leq 0, \tag{3.33}$$

$$\sum_{\emptyset \neq J \subseteq I} \omega(\vee_J) \leq \sum_{\emptyset \neq J \subseteq K} \omega(\vee_J). \tag{3.34}$$

Corollary 23. *Every nontrivial binary weighted clone over the clone* $\vee P_{01}$ *is either* $\mathbf{BP}(W_\vee)$ *or is equal to* $W(M)$ *for some* M *such that:*

(1) $M \subseteq \mathbb{Q}^{t=1}_{\geq 0}$ *and satisfies Property (∗∗) or*

(2) $M \subseteq \mathbb{Q}^{t=0}_{\geq 0}$ *is contained in the square with vertices* $(1, 0, 1), (0, 0, 1), (0, 1, 1), (1, 1, 1)$ *and satisfies Property (∗∗) or*

(3) $M \subseteq \mathbb{Q}^3_{\geq 0}$ *is contained in the cube with vertices* $(1, 0, 0), (0, 0, 0), (0, 1, 0), (1, 1, 0), (1, 0, 1), (0, 0, 1), (0, 1, 1), (1, 1, 1)$ *and satisfies Property (∗∗).*

For every such set M $W(M) \subsetneq \mathbf{BP}(W_\vee)$ *if and only if* $M \subseteq \mathbb{Q}^{t=1}_{\geq 0}$*. For every two such sets* M*,* M' $W(M) \subseteq W(M')$ *if and only if* $M \subseteq M'$*.*

Conclusion

In this thesis, we introduced the concept of binary weighted clones and characterized all binary weighted clones and some particular weighted clones over certain clones on Boolean domain, namely $\wedge P_0$, $\wedge P_1$, $\wedge P_{01}$ and dually $\vee P_0$, $\vee P_1$, $\vee P_{01}$. However, a complete description of all weighted clones remains widely open.

We believe that partial description of weighted clones over the clones AP_0 and AP_1 is possible by using the approach of this thesis.

For richer clones, e.g. MP, MP_0^∞, MP_1^∞, more sophisticated methods seem to be necessary.

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